MATHEMATICS OF ANNUITIES AND INSURANCE

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Edwards Bros.-----Publishers,

Ann Arbor, Mich.

MATHEMATICS OF ANNUITIES AND INSURANCE

LESSON I.

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ELEMENTS OF ACTUARIAL THEORY.

Preliminary definitions.

Interest may be defined as the consideration for the use of capital. The capital, that which is invested, and the interest, that which is earned by the capital, need not be of the same kind. For example, the capital may be land and the interest may be wheat paid for the use of the land. It is usual, however, to express both capital and interest in terms of one and the same thing, generally some monetary unit, as the dollar, pound, franc, mark. The invested capital is called the Principal. We have next to consider the rate at which a given capital is earning interest and this notion involves the idea of time. We therefore select some interval as the unit of time. The unit of time almost invariably employed in the theory of interest is the year. It is clear that interest when received may be added to the capital and so in turn earn interest. This process is called compounding. Interest is usually compounded at the end of stated intervals, as every three months, six months, year. We are now prepared to define what is meant by "The Rate of Interest."

(a) The Effective Rate of Interest.

The total interest earned on one unit of principal (one dollar) in one unit of time (one year) when interest is compounded at the end of each stated interval is called the effective

rate of interest.

(b) The Nominal Rate of Interest.

The total interest earned on one unit of principal (one dollar) in one unit of time (one year) when interest is not compounded at the end of each stated interval is called the nominal rate of interest.

From the above definitions it follows that the nominal and effective rates of interest will coincide when, and only when, the stated interval happens to be the unit of time (one year).

In commercial transactions the rate of interest is usually quoted as a rate per cent. (that is, per centum, or per hundred units of principal instead of at the rate per unit of principal, as in the above definitions). To find the rate as above defined, therefore, it will only be necessary to divide the commercial rate by 100. For example, the mathematical rate corresponding to the commercial rate 6% is 6/100, or .06.

Before enlarging upon and illustrating the foregoing notions of effective and nominal rates of interest, we shall take up the theory of Simple Interest. In this theory, interest is not compounded, and consequently the rate of interest coincides with the above defined nominal rate of interest. For example, if it is agreed to pay one cent per month for the use of each dollar of principal, interest not to be added to principal, the total interest earned in one year would be twelve cents for each dollar, that is, .12 of a dollar. The rate of interest is therefore .12 or 12%. Similarly, one cent per dollar every three months would be

equivalent to a rate of .04 or 4%. To deduce the formulas of simple interest let

P = The Principal.

S = The <u>Amount</u> to which that principal will accumulate.

n = The <u>Term</u>, or the number of years the principal is under investment.

i = The Rate of Interest, or the interest on 1 for one year. Since, by definition, the interest on 1 for one year is i, the interest on 1 for n years would be n times i, that is, ni. Again, if ni is the interest earned in n years by one unit (or dollar), P times ni, or Pni, would be the interest earned by P units (or dollars) in n years. The amount, S, is the sum of the original principal and the interest earned in n years, hence

$$S = P + Pni = P(1 + ni) \tag{1}$$

This formula leads to the following rule:--Multiply the interest on one dollar for one year by the number of years, add 1 to the product, and multiply the sum by the principal.

Example.--A agrees to lend B the sum of $4368 at an annual interest of 6%, interest and principal to be paid at the end of eight years. What sum must B pay? In this problem we have $P = 4368$, $i = .06$, $n = 8$. Hence $ni = .48$, $(1 + ni) = 1.48$, and $S = 4368$ times $1.48 = 4368(1.48) = 6464.64$. The total sum is therefore $6464.64.

Formula (1) contains in itself the whole theory of simple interest and by properly applying it any problem in that subject

may be solved. The formula involves four quantities, P, S, n, i, and if any <u>three</u> of these are given the fourth may be determined by the laws of algebra. In the above example P, n, and i were given to determine S. Suppose that S, n and i are given to determine P. Dividing both members of formula (1) by (1 + ni) we have

$$P = S / (1 + ni) \qquad (2)$$

Rule. Multiply the interest on one dollar for one year by the number of years, add 1 to the product, and divide the amount by the sum.

Example.--$5678 was paid for the loan of a certain sum of money at 7% per annum for five years and six months, simple interest. What was the sum? Solution. S = 5678, n = 5.5, i = .07, hence, substituting in (2), we have

$P = 5678 / (1 + 5.5 \times .07) = 5678 / (1 + .385) = 5678 / 1.385 = 4099.64$. The sum is thus seen to be $4099.64.

Let P, S and i be given to find n. We have, starting with (1)

$$S = P + Pni$$

by transposition, $S - P = Pni$

dividing both members by Pi we arrive at the following result:

$$n = (S - P) / Pi \qquad (3)$$

Rule.--Multiply the interest on one dollar for one year by the principal, and divide the difference between the amount and the principal by the product.

Example.--In how many years will $5764.16 amount to $6754.78 at 4% simple interest? Here P = 5764.16, S = 6754.78, i = .04. Substituting in (3) we have

$n = (6754.78 - 5764.16) / 5764.16 \times .04 = 990.62 / 230.5664 =$
$4.296+ = 4$ years, 3 months, ~~106~~ 17 days.

This is on the assumption that there are 360 days in one year.

Finally, it may be required to find the rate of interest, i, when S, P and n are given. In obtaining formula (3) we found that $S - P = Pni$. Dividing both members of this equation by Pn we have

$$i = (S - P) / Pn \qquad (4)$$

Rule.--Divide the difference between the amount and the principal by the product of the principal and number of years.

Example.--At what rate of interest will \$4632 amount to \$5378.65 in 2 years and 72 days? In this problem $P = 4632$, $S = 5378.65$, $n = 2.2$, hence by (4)

$$i = (5378.65 - 4632) / 4632 \times 2.2 = 746.65 / 10190.4 = .07327.$$

The rate of interest is therefore 7.237%.

The total interest earned by a principal P in n years at the rate of interest i we have seen to be Pni. This quantity, which is evidently the difference between the amount and the principal, we shall designate by the symbol I. This leads to the formula

$$I = S - P = niP \qquad (5)$$

Questions and Problems in the Theory of Simple Interest.

The questions and problems here proposed are to be answered by the student.* In solving problems use plenty of paper, write

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* In case a * precedes the question or problem it is to be answered or worked in full and returned within ten days to me for correction and criticism. In answering questions, imagine the instructor present, propose the question to yourself orally and answer it in the same way without reference to the text in the lesson.

at the head the formula to be used in the solution, carry out the arithmetic work in full, neatly and systematically, and take care that the decimal points are in their proper places. Always go over your work a second time, if possible in some new way, in order to get a check on the first solution.

1. Define interest. What is meant by capital? May capital be a commodity? May interest be a commodity? How are they usually measured?

2. Define principal. What is the unit of time employed in the theory of interest? Would it be possible to select another unit of time?

3. What is meant by compounding interest? Does the stated interval in compounding necessarily coincide with the unit interval?

4. Define effective rate of interest. A agrees to pay B five cents at the end of every six months for the use of one dollar. If interest is compounded what will be the effective rate?

5. Define nominal rate of interest. If interest is not compounded in the above transaction what will be the nominal rate? Does the nominal rate of interest ever coincide with the effective rate?

6. What is rate of interest per cent. and how is it related to the mathematical rate of interest? If two cents is paid every month for the use of three dollars what will be the nominal rate of interest per cent.?

*7. What is the total interest earned by $857.44 in twelve years at 6.25%?

*8. $3478 accumulates at 5% simple interest to $4125 in how many years? 3 yr 8 mo 19 da

*9. Eight years, three months ago, $4728 was put out at five and one half per cent. simple interest. What is the amount now? $6873.33

*10. The interest earned on $6579 in four years and two months was $2345, what was the rate of interest? .0855

*11. What capital must be employed at six per cent. simple interest for ten years to earn $888? $1480

*12. How long will it take for a given principal to double itself at simple interest at rate i?

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LESSON II.

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Theory of Discount.--Discount may be defined as the consideration for the immediate payment of a sum due at a future date. The sum due will be denoted by S', the present value of the sum by P' and the discount by D. If, for example, for the present payment of a note for \$100 due three months hence, \$98 is accepted, the discount is \$2. Here S' = 100, P' = 98, D = 2. A sum due may be discounted successively and this process is called compound discounting. Example.--To determine the present value of \$100 due in six months when it is agreed to discount every three months, the discount being 2 cents for the present payment of each dollar of the sum due for each stated interval, in this case three months, of discounting.

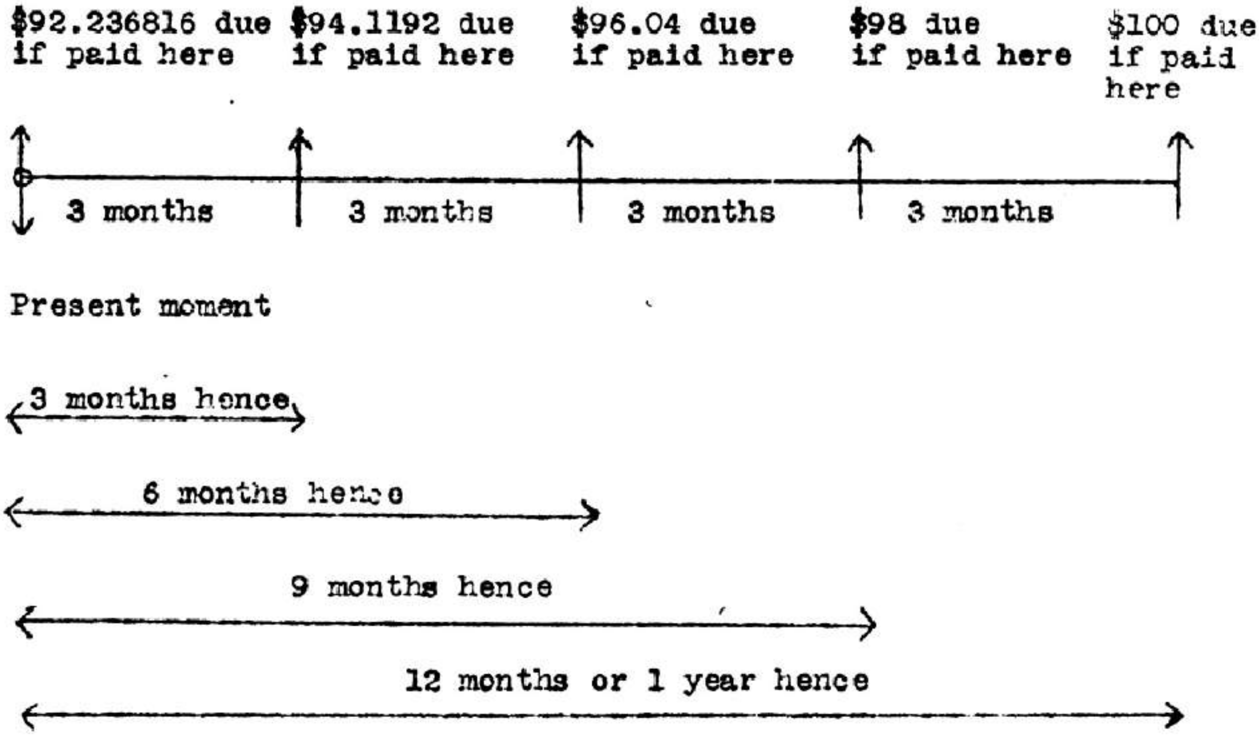
1st discounting.....S' = 100, hence D = 2, hence P' = 98

2d " S' = 98, " D = 1.96, " P' = 96.04.

The total discount is thus seen to be 100 - 96.04 = 3.96. Hence D = \$3.96.

Example.--For the present payment of \$100 due one year hence it is agreed to discount every three months, two cents to be allowed for the payment at the beginning of each interval for each dollar due at the end of the interval. What is the total discount?

The diagram below illustrates graphically the process of compound discounting in this example.



1st discounting	S' = 100.00	hence D = 2.00	P' = 98.00
2d "	S' = 98.00	" D = 1.96	P' = 96.04
3d "	S' = 96.04	" D = 1.9208	P' = 94.1192
4th "	S' = 94.1192	" D = 1.882384	P' = 92.236816

hence the total discount which is the difference between the sum due $100 and the final present value $92.236816 is equal to $7.763184. This result may also be obtained by adding up the several discounts.

We are now prepared to define what is meant by "The Rate of Discount."

(a) The Effective Rate of Discount.

The total discount on one unit (one dollar) of the sum due

in one unit (one year) of time when discount is compounded at the beginning of each stated interval is called the effective rate of discount.

(b) The Nominal Rate of Discount.

The total discount on one unit (one dollar) of the sum due in one unit (one year) of time when the discount is not compounded at the beginning of each stated interval is called the nominal rate of discount.

From the above definitions it follows that the nominal and effective rates of discount will coincide when, and only when, the stated interval happens to be the unit of time (one year).

In commercial transactions the rate of discount is usually quoted as a rate per cent. (that is, per centum, or per hundred) units of the sum due instead of at the rate per unit of the sum due, as in the above definitions. To find the rate as above defined, therefore, it will only be necessary to divide the commercial rate by 100. For example, the mathematical rate corresponding to the commercial rate 5% is 5/100, or .05.

Before enlarging upon the foregoing notions of effective and nominal rates of discount, we shall take up the theory of Simple Discount. In this theory discount is not compounded and consequently the rate of discount coincides with the above defined nominal rate of discount. For example, if it is agreed to pay one cent per month for the immediate payment of each dollar of the sum due, discount not to be taken from the sum due at each successive monthly discounting, the total discount earned in one year would be twelve cents for each dollar of the sum due, that is,

.12 of a dollar. The rate of discount is therefore .12 or 12%. Similarly, one cent per dollar every three months would be equivalent to a rate of .04 or 4%. To deduce the formulas of simple discount let

S' = The Sum Due.

P' = The Present Value (immediate payment) to which the sum due is discounted.

n = The Term, or the number of years the sum due is discounted.

d = The Rate of Discount, or the discount on 1 for one year.

Since, by definition, the discount on 1 for one year is d, the discount on 1 for n years would be n times d, that is dn. Again, if dn is the discount in n years on one unit (or dollar), S' times dn, or $S'dn$, would be the discount on S' units (or dollars) in n years. The present value, P', is the sum due (original) less the discount earned in n years, hence we derive the formula

$$P' = S' - S'dn = S'(1 - dn) = S'(1 - nd) \qquad (1')$$

Example.--A note for \$5362 due in four years is paid now, the rate of discount being 7%. What is the present value of the note?

Solution.--Here $S' = 5362$, $n = 4$, $d = .07$. Hence, by formula (1') $P' = 5362(1 - .07 \times 4) = 5362 \times .72 = 3860.64$. The present value of the note is therefore \$3860.64.

Formula (1') contains in itself the whole theory of simple discount and by properly applying it any problem in that subject may be solved. The formula involves four quantities, S', P', n, d, and if any <u>three</u> of these are given the fourth may be determined by the laws of algebra. Solving in succession for S', n and d,

we have the following formulas:

$$S' = P' / (1 - nd) \qquad (2')$$

$$n = (S' - P') / S'd \qquad (3')$$

$$d = (S' - P') / S'n \qquad (4')$$

*It is left as an exercise for the student to derive these results. The total discount earned by a sum due, S', in n years at the rate of discount d we have seen to be $S'nd$. This quantity, which is evidently the difference between the sum due and the present value, we shall denote by the symbol D. This leads to the formula $D = S' - P' = ndS'$ (5')

Examples.

*1. A bill of $8765 brought $8428 when discounted at the rate 3%, what was the time for which it was discounted?

*2. If the preceding bill had been for ninety days what would have been the rate of discount?

*3. A six month bill is purchased for $3648 at a rate of discount 3 and 5/8 per cent, what was the face of the bill?

*4. Make up one example for each of the five formulas in lesson two and solve completely in each case.

*5. How long would it take for a given sum due to yield a present value of 1/2 the sum due at a given rate of discount d? Of 1/kth?

6. Note carefully the similarity in the development of the theory of simple interest and the theory of simple discount. Have the two theories been brought into any definite relation to each other as yet? Has any relation, for example, been established

between the rate of interest i and the rate of discount d?

7. Can you name any usual form of commercial transaction or financial transaction which is based on a rate of discount instead of a rate of interest? Are any rates of discount quoted in the financial journals? What in your mind is the chief distinction between a rate of interest and a rate of discount?

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L E S S O N III.

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We have seen in the preceding lesson that the present value of 5362 for four years at the rate of discount 7% is 3860.64. Conversely, suppose this sum received, 3860.64, is at once put out at simple interest for four years at the rate or interest 7%. What will be its amount at the end of that time? Here we have a problem in simple interest and by formula (1)

$$S = 3860.64(1 + 4 \times .07) = 4941.62.$$

This falls short of 5362 by 420.38 and it is clear that there is a great difference between a rate of discount of 7% and a rate of interest of 7%. Now we call that rate of interest, i, which, when applied to 3860.64, as principal, for four years, will amount to the sum due, 5362, the rate of interest corresponding to the rate of discount 7%. The rate of discount is also said to correspond to the rate of interest, i. Let us find the rate i in this problem corresponding to $d = .07$. By formula (4) we have

$$i = (5362 - 3860.64) \ / \ (3860.64 \times 4) = .097222+$$

Hence $i = 9.722\%$ when $d = 7\%$ and $n = 4$. To show that this is the correct value of i, by (1) we have,

$$S = 3860.64(1 + .09722 \times 4) = 3860.64 \times 1.38889 = 5362.$$

We now proceed to establish in general a relation between the rate of interest and the corresponding rate of discount by bringing our formulae into agreement with business practice. We define the rate of interest i corresponding to the rate of discount d as that

rate of interest which in n years will increase a principal P to an amount S where S is a sum due in n years which at the rate of discount d will have a present value P. By formula (1) we have $S / P = 1 + ni$ and by (1') $S' / P' = 1 / (1 - nd)$; but by our definition $S' = S$ and $P' = P$, hence

$$1 + ni = 1 / (1 - nd)$$

$$(1 + ni)(1 - nd) = 1$$

$$i - d - ndi = 0.$$

Solving for i and d respectively, we have

$$i = d / (1 - nd) \qquad (6')$$

$$d = i / (1 + ni) \qquad (7')$$

In the special case when the time is one year, that is, when n = 1, the two preceding formulas become:

$$i = d / (1 - d) \qquad (8')$$

$$d = i / (1 + i) \qquad (9')$$

Example.--A bank discounts a bill due in one year at the rate of discount 6%; what is the rate of interest realized by the bank on this transaction?

Substituting in formula (8') $i = .06 / (1 - .06) = .06 / .94 = .06\frac{36}{94}$. Hence $i = 6\frac{18}{47}\%$. It thus appears that the rate of interest is more than one third of one per cent higher than the corresponding rate of discount.

* The Bank of England Rate of Discount is quoted at 3 and 5/8 per cent; what is the rate of interest?

Reviewing what has gone before we now see that the chief distinction between a rate of interest i and a rate of discount d is that the ~~former~~ is based upon a sum due in the future while the ~~latter~~ former is based on the principal, a sum in hand.

Relations Connecting Effective and Nominal Rates of Interest.

With respect to any given transaction there is an effective rate of interest i and a corresponding nominal rate of interest j; the relation between these quantities can be expressed by an algebraic formula. This formula will involve m, the number of stated intervals in one year. Since the nominal rate is j, it follows that during each stated interval 1/mth of a year in length, one unit would accumulate j/m in interest which added to the unit gives an amount 1 + j/m. If the principal 1 amounts in one interval to (1 + j/m) it follows by proportion that a principal of P would amount in one interval to P(1 + j/m). This gives us the following rule: to find the amount of a principal P at the end of 1/mth of a year at the nominal rate of interest j, multiply the principal by (1 + j/m). To find the effective rate of interest i we must by definition compound or add the interest to the principal at the end of every stated interval. This leads to the following scheme:

(1) Number of Interval.	(2) Principal at beginning of Interval.	(3) Interest during Interval.	(4) = (2) + (3) Amount at end of Interval.
1	1	j/m	$(1 + j/m)$
2	$(1 + j/m)$	$(1 + j/m)j/m$	$(1 + j/m)^2$
3	$(1 + j/m)^2$	$(1 + j/m)^2 j/m$	$(1 + j/m)^3$
.	.	.	
.	.	.	
m	$(1 + j/m)^{m-1}$	$(1 + j/m)^{m-1} j/m$	$(1 + j/m)^m$

This shows that at the end of the mth interval the amount is $(1 + j/m)^m$. But we started out with 1 unit hence the total interest earned in m intervals or one year is the difference between these quantities or

$$(1 + j/m)^m - 1$$

This by definition is equal to the effective rate of interest i, hence we arrive at the fundamental formula:

$$i = (1 + j/m)^m - 1 \qquad (6)$$

Transposing, we have the useful relation

$$1 + i = (1 + j/m)^m \qquad (7)$$

Taking the mth root of both sides, transposing and solving for j we have

$$j = m\left[(1 + i)^{1/m} - 1\right] \qquad (8)$$

*The student should carry out the solution indicated in full. The number of times per annum that interest is added is m and this is often called the <u>frequency of conversion</u>. The nominal rate j with a bracketed subscript to the right, $j_{(m)}$, means and should be read "a nominal rate of interest j convertible m times per annum."

Example.--At a certain bank the rate of interest on deposits is 3% and interest is added to principal every six months; find i, j, m. Here the nominal rate $j = .03$, $m = 2$, and by substituting in (6) we have $i = (1 + .03/2)^2 - 1 = (1.015)^2 - 1 = .030225$. Hence $i = 3.0225\%$. It is thus seen that the effective rate is slightly higher than the corresponding nominal rate convertible half-yearly or twice per annum.

Example.--Given the effective rate of interest 3%, what is the corresponding nominal rate; (a) when interest is convertible

semi-annually, (b) when interest is convertible quarterly? Here i and m are given to find j, hence we substitute in formula (8) and have

(a) $j = 2((1 + .06)^{1/2} - 1) = 2(1.06)^{1/2} - 2 = 2.0591 - 2 = .0591.$

(b) $j = 4((1 + .06)^{1/4} - 1) = 4(1.06)^{1/4} - 4 = 4.0587 - 4 = .0587.$

In this problem it is necessary to extract the square and fourth roots of 1.06. The final result shows that in case (a) $j = 5.91\%$ and in case (b) $j = 5.87\%$. We note again that the nominal rate is smaller than the corresponding effective rate, also that if the effective rate i remains the same the corresponding nominal rate will vary with m; when m increases j decreases. In general, when i is constant, j grows smaller as the frequency of conversion increases. In the above example, when $i = .06$, j diminishes from 5.91% to 5.87% as the frequency of conversion increases from twice a year to four times a year.

*It is left as an exercise for the student to investigate, when j is constant, the variation of i as the frequency of conversion is increased.

Suggestion.--Take $j = .05$ and compute the value of i when m = 1, 2, 3, 4, etc., using formula (6).

The Theory of Compound Interest.

To find the amount of 1 in n years at compound interest. Let the effective rate of interest be i. At the end of the first year we have $1 + i$ and the unit has been increased in the ratio of 1 to $(1 + i)$. Similarly during the second year the principal $(1 + i)$ will be increased in the ratio of 1 to $(1 + i)$ and will

therefore amount at the end of the year to $(1 + i)(1 + i)$ or $(1 + i)^2$. Proceeding in this way we find that at the end of n years the amount is $(1 + i)^n$.

*The student is requested to make out a schedule proof of the above result such as that on page ~~12.~~ 16

Let P be the principal and S the amount of P at compound interest at the effective rate i for n years; let I be the interest earned by P in that time. Since 1 amounts to $(1 + i)^n$ in n years it follows that P would amount to $P(1 + i)^n$. We have, therefore, the

formula $S = P(1 + i)^n$ present value of due sum (9)

Hence $P = S / (1 + i)^n = Sv^n$ (10)

where $v = 1 / (1 + i)$ (11)

If we replace in the above formulae $1 + i$ by $(1 + j/m)^m$, to which it is equivalent according to formula (7), we have

$$S = P(1 + j/m)^{mn} \qquad (9")$$

$$P = S / (1 + j/m)^{mn} = Sv'^{mn} \qquad (10")$$

where $v' = 1 / (1 + j/m)$ (11")

These formulae enable us to express the relation between P and S in terms of the nominal rate j and the frequency of conversion m.

*It is left for the student to find the value of I, (a) in terms of i, n, P; (b) in terms of j, n, m, p. Also, (c) in terms of i, n, S; (d) in terms of j, n, m, S. Tables of interest are published giving the values of the functions $(1 + i)^n$ and v^n for various rates of interest and number of intervals or years. In solving any problem use should be made of these tables to abridge the work as far as possible.

Example.--What will be the amount of $6475 at 4% compound interest in 9 yrs.

By (9) we have $S = 6475(1 + .04)^9 = 6475 \times 1.4233 = 9215.87$.

The value of 1.04^9 was taken from an interest table. The amount then is $9215.87.

Example.--$3674 is placed in a bank; what will it amount to in 25 years if interest is 3% and compounded semi-annually?

Here we have to do with a nominal rate of 3% convertible twice a year, hence we use formula (9") with $j = .03$ and $m = 2$. Substituting we have:

$S = 3674(1 + .03/2)^{2x25} = 3674 \times 1.015^{50} = 3674 \times 2.1052 = 7734.50$.

Hence the amount at end of 25 years will be $7734.50.

Example.--What principal invested for 21 years at three and one half per cent compound interest will amount to ten thousand dollars?

Here $S = 10000$, $i = .035$, $n = 21$; substituting in (10) we have $P = 10000/1.035^{21} = 10000 \times .48557 = 4855.70$. The required principal is therefore $4855.70. In this problem $v = 1/1.035$ and v^{21} was taken from the interest table directly.

Example.--What sum will accumulate to $8766 in 18 years when interest is 5% compounded semi-annually?

In this problem $S = 8766$, $j = .05$, $m = 2$, $n = 18$; formula (10") therefore gives $P = 8766(1/1.025)^{36} = 8766 \times .41109 = 3603.61$. Hence the sum is $3603.61.

*Find the effective rate corresponding to a nominal rate of 5% convertible (a) half yearly (b) quarterly.

*Find the nominal rate (a) convertible twice a year (b) four times a year corresponding to an effective rate of 4%.

*State and work out in full four examples illustrating formulas 9, 10, 9", 10", using your interest tables in the solution.

*Work out the rate of discount corresponding to the following rates of simple interest: 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5 per cent.

*Work out the rates of interest corresponding to the following rates of simple discount: 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5 per cent.

*Solve formula (9) for i in terms of S, P and n.

*Solve formula (9") for j in terms of S, P, m, and n.

*$5632 after three years at compound interest amounts to $7642, what was the rate of interest?

*Add the geometric series in column (3) on page 16 and show that its sum, which is the total interest earned by the unit during the year, is the same as that given by formula (6).

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L E S S O N IV.

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Referring to the preceding lesson, formulas 9 and 9", we see that

$$i = (S/P)^{1/n} - 1 \qquad (10a)$$

$$j = m((S/P)^{1/mn} - 1) \qquad (11a)$$

$$I = S - P = P((1 + i)^n - 1) \qquad (12)$$

$$= S(1 - v^n) \qquad (13)$$

$$= P((1 + j/m)^{mn} - 1) \qquad (14)$$

$$= S(1 - 1 / (1 + j/m)^{mn}) \qquad (15)$$

$$= P((1 + i')^{mn} - 1) \qquad (16)$$

$$= S(1 - v'^{mn}) \qquad (17)$$

where $i' = j/m$ and $v' = 1 / (1 + i')$.

Formulas (16) and (17) suggest that we find the values of $(1 + j/m)^{mn}$ and $1 / (1 + j/m)^{mn}$ in (14) and (15) by entering the interest tables with the argument $i'\% = j/m$ for $(1 + i')^{mn}$ and v'^{mn}.

Example.--What is the interest on \$2645 for 16 years at 5% convertible half yearly? By (16)

$$I = ((1.025)^{32} - 1) \times 2645 = 2645 \times 1.2038 = 3184.05.$$

Hence I = \$3184.05. Here 1.025^{32} was found by entering the table $(1 + i)^n$ with $i = 2\frac{1}{2}\%$ and $n = 32$.

Example.--A certain sum at 5% convertible semi-annually amounts in twenty years to \$8466, find the interest. Here by (17),

$$I = 8466(1 - 1/1.025^{40}) = 8466 \times .6276 = 5313.26.$$ Hence

I = \$5313.26. To find $1/1.025^{40}$ we enter the table v^n with

$i\% = 2\frac{1}{2}\%$ and $n = 40$.

***<u>For students who have studied logarithms.</u>

It will be noticed that (9) involves four quantities: S, P, i, n. Any one can be found in terms of the remaining three and formulas have already been derived for S, P, and i. To find n, however, requires the use of logarithms. The solution is as follows: Taking the logarithm of each member of (9) and remembering that the logarithm of the product is equal to the sum of the logarithms of the factors, we have

$$\log S = \log P + \log (1 + i)^n$$
$$= \log P + n \log (1 + i)$$

since the logarithm of a power is equal to the product of the power and the logarithm of the base. Solving for n we have

$$n = (\log S - \log P) \ / \ \log (1 + i) \qquad (18)$$

Example.--In how many years will \$4364 at 5% amount to \$6849?

$$\begin{aligned} \log 6849 &= 3.835627 \\ \log 4364 &= 3.639885 \\ & 0.195742 \\ \log 1.05 &= 0.021189 \qquad \text{hence } n = 9.2379. \end{aligned}$$

The term is therefore 9 years, 2 months, 26 days. In practice the above interest of \$2485 would be considered as made up of two parts, compound interest for an integral number of years (or intervals in case interest is compounded more than once a year), 9 in this case, and simple interest for the remaining fractional part of a year.

*It is left to the student to solve the above problem on the latter basis.

Relations Between Effective and Nominal Rates of Discount.

With respect to any given transaction in compound discount there is an effective rate of discount d, and a corresponding nominal rate of discount f; the relation between these quantities can be expressed by an algebraic formula. This formula will involve m, the number of stated intervals in one year (see definitions, pages 9 and 10). Since the nominal rate is f, it follows that during each stated interval 1/mth of a year in length, the discount on one unit of the sum due at the end of the interval would be f/m, which taken from the unit leaves 1 - f/m as the value of the same unit if paid at the beginning of the interval. If the sum due, 1, is discounted in one interval to 1 - f/m it follows by proportion that a sum due of S' would be discounted in one interval to S'(1 - f/m). This gives us the following rule: to find the value of a sum due, S', at the beginning of an interval of 1/mth of a year at the nominal rate of discount f, convertible m times a year, multiply the sum due by (1 - f/m). To find the effective rate of discount d we must by definition compound or subtract the discount from the sum due at the beginning of every stated interval. This leads to the following scheme:

(1) Number of Interval.	(2) Sum due at end of Interval.	(3) Discount during Interval.	(4)=(2)-(3) Value of sum due at beginning of Interval.
m	1	f/m	$1 - f/m$
$m - 1$	$(1 - f/m)$	$(1 - f/m)f/m$	$(1 - f/m)^2$
$m - 2$	$(1 - f/m)^2$	$(1 - f/m)^2 f/m$	$(1 - f/m)^3$
"	"	"	"
"	"	"	"
2	$(1 - f/m)^{m-2}$	$(1 - f/m)^{m-2} f/m$	$(1 - f/m)^{m-1}$
1	$(1 - f/m)^{m-1}$	$(1 - f/m)^{m-1} f/m$	$(1 - f/m)^m$

This schedule shows that at the beginning of the first interval, that is, the present moment, the value of the unit due m intervals or one year hence is $(1 - f/m)^m$. Hence the total discount for the year is $1 - (1 - f/m)^m$. This by definition is equal to the effective rate of discount d, hence we have the formula:

$$d = 1 - (1 - f/m)^m \tag{18a}$$

This leads to the relations,

$$(1 - f/m)^m = 1 - d \tag{19}$$

$$f = m(1 - (1 - d)^{1/m}) \tag{20}$$

The student should carry out the solution indicated in full. It is to be noted that d, the total discount on the unit for the year, could be obtained from the above schedule by adding the geometric series in the third column.

*This is left as an exercise for the student. The number of times per annum that discount is deducted is m and this is often called the frequency of conversion. The nominal rate of discount f with a bracketed subscript to the right, $f_{(m)}$, means and should

be read "a nominal rate of discount f convertible m times per annum."

Example.--The Bank of England discount rate on three month bills is 3 1/2%. What is the corresponding effective rate of discount?

Here $f = .035$, $m = 4$. Substituting in formula (18a) we have $d = 1 - (1 - .035/4)^4$. Hence $d = 1 - .99125^4$. Taking the fourth power of .99125 by logarithms or otherwise we have $d = 1 - .965455 = .034545$. Therefore the rate of discount is 3.4545%, that is, somewhat less than the nominal rate.

Example.--Given the effective rate of discount 6%, what is the corresponding nominal rate; (a) when discount is convertible semi-annually, (b) when discount is convertible quarterly? Here d and m are given to find f. By (20) we have

(a) $f = 2(1 - .94^{1/2}) = 2(1 - 0.96954) = .06092$. Hence $f = 6.092\%$.

(b) $f = 4(1 - .94^{1/4}) = 4(1 - 0.98465) = .06140$. Hence $f = 6.140\%$.

We see that the nominal rate is larger than the corresponding effective rate, also that if the effective rate d remains fixed the corresponding nominal rate will vary with m; when m increases f increases. In general, when d is constant, f grows larger as the frequency of conversion increases. In the above example, when $d = .06$, f increases from 6.09% to 6.14% as the frequency of conversion increases from twice a year to four times a year.

*It is left as an exercise for the student to investigate, when f is constant, the variation of d as the frequency of conversion is increased.

Suggestion.--Take $f = .05$ and compute the value of d when $m = 1, 2, 3, 4$, etc., using formula (18a).

The Theory of Compound Discount.

It is left for the student to elaborate the theory along the same general lines as that of compound interest on pages 18 and 19. Let S' be the sum due and P' its present value at compound discount at the effective rate d, S' being due in n years; let D be the discount on S'. The resulting formulas are:

$$P' = S'(1 - d)^n = S'(1 - f/m)^{mn} \qquad (21)$$

$$S' = P' \,/\, (1 - d)^n = P' \,/\, (1 - f/m)^{mn} \qquad (22)$$

$$D = S' - P' = S'(1 - (1 - d)^n) = S'(1 - (1 - f/m)^{mn})$$

$$= P'(1 \,/\, (1 - d)^n - 1) = P'(1 \,/\, (1 - f/m)^{mn} - 1)$$

*In developing this theory make out a schedule similar to that on page 25 showing the discount for each year and that the total discount for the n years at the effective rate d is $1 - (1 - d)^n$ for each unit of the sum due. Also find the total discount for the n years by adding up the geometric series in the third column of your schedule.

Correlation of the theory of interest with the theory of discount.--It should be carefully noted that the theory of compound interest and the theory of compound discount have been independently developed side by side. We shall now correlate or bring them into correspondence. In doing this we shall be guided, of course, by business practice. Referring to the definition of correspondence given on page 14 we have S/P = S'/P'. Hence by (9), (9"), (21), we have

$$S/P = (1 + i)^n = (1 + j/m)^{mn} = S'/P' = 1 \,/\, (1 - d)^n = 1 \,/\, (1 - f/m)^{mn}.$$

Hence $$1 + i = 1 / (1 - d) = (1 + j/m)^m = 1 / (1 - f/m)^m \qquad (23)$$

Equations (23) are the fundamental relations connecting the <u>corresponding</u> effective and nominal rates of interest and discount and the frequency of conversion. It will be noticed that there are five quantities: i, j, d, f, m, and the above relations connecting them are independent of n. These five quantities are usually taken together with a sixth, $v = \frac{1}{1 + i}$, and the above formulas make it possible to express any one of the six in terms of one or more of the others.

*Show that $i - d = id$; that $j - f = jf/m$; that $d = iv$.

*Express i in terms of f and m. Express d in terms of j and m.

*Express j in terms of d and m. Express j in terms of f and m. Of v and m.

*Express f in terms of v and m. Express d in terms of v.

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T A B L E I.

Amount of 1: viz., $(1 + i)^n$.

n	$2\frac{1}{2}$%	3%	$3\frac{1}{2}$%	4%	$4\frac{1}{2}$%	5%	n
1	1.02500	1.03000	1.03500	1.04000	1.04500	1.05000	1
2	1.05063	1.06090	1.07123	1.08160	1.09203	1.10250	2
3	1.07689	1.09273	1.10872	1.12486	1.14117	1.15763	3
4	1.10381	1.12551	1.14752	1.16986	1.19252	1.21551	4
5	1.13141	1.15927	1.18769	1.21665	1.24618	1.27628	5
6	1.15969	1.19405	1.22926	1.26532	1.30226	1.34010	6
7	1.18869	1.22987	1.27228	1.31593	1.36086	1.40710	7
8	1.21840	1.26677	1.31681	1.36857	1.42210	1.47746	8
9	1.24886	1.30477	1.36290	1.42331	1.48610	1.55133	9
10	1.28008	1.34392	1.41060	1.48024	1.55297	1.62889	10
11	1.31209	1.38423	1.45997	1.53945	1.62285	1.71034	11
12	1.34489	1.42576	1.51107	1.60103	1.69588	1.79586	12
13	1.37851	1.46853	1.56396	1.66507	1.77220	1.88565	13
14	1.41297	1.51259	1.61869	1.73168	1.85194	1.97993	14
15	1.44830	1.55797	1.67535	1.80094	1.93528	2.07893	15
16	1.48451	1.60471	1.73399	1.87298	2.02237	2.18287	16
17	1.52162	1.65285	1.79468	1.94790	2.11338	2.29202	17
18	1.55966	1.70243	1.85749	2.02582	2.20848	2.40662	18
19	1.59865	1.75351	1.92250	2.10685	2.30786	2.52695	19
20	1.63862	1.80611	1.98979	2.19112	2.41171	2.65330	20
21	1.67958	1.86029	2.05943	2.27877	2.52024	2.78596	21
22	1.72157	1.91610	2.13151	2.36992	2.63365	2.92526	22
23	1.76461	1.97359	2.20611	2.46472	2.75217	3.07152	23
24	1.80873	2.03279	2.28333	2.56330	2.87601	3.22510	24
25	1.85394	2.09378	2.36324	2.66584	3.00543	3.38635	25
26	1.90029	2.15659	2.44596	2.77247	3.14068	3.55567	26
27	1.94780	2.22129	2.53157	2.88337	3.28201	3.73346	27
28	1.99650	2.28793	2.62017	2.99870	3.42970	3.92013	28
29	2.04641	2.35657	2.71188	3.11865	3.58404	4.11614	29
30	2.09757	2.42726	2.80679	3.24340	3.74532	4.32194	30
31	2.15001	2.50008	2.90503	3.37313	3.91386	4.53804	31
32	2.20376	2.57508	3.00671	3.50806	4.08998	4.76494	32
33	2.25885	2.65234	3.11194	3.64838	4.27403	5.00319	33
34	2.31532	2.73191	3.22086	3.79432	4.46636	5.25335	34
35	2.37321	2.81386	3.33359	3.94609	4.66735	5.51602	35
36	2.43254	2.89828	3.45027	4.10393	4.87738	5.79182	36
37	2.49335	2.98523	3.57103	4.26809	5.09686	6.08141	37
38	2.55568	3.07478	3.69601	4.43881	5.32622	6.38548	38
39	2.61957	3.16703	3.82537	4.61637	5.56590	6.70475	39
40	2.68506	3.26204	3.95926	4.80102	5.81636	7.03999	40
41	2.75219	3.35990	4.09783	4.99306	6.07810	7.39199	41
42	2.82100	3.46070	4.24126	5.19278	6.35162	7.76159	42
43	2.89152	3.56452	4.38970	5.40050	6.63744	8.14967	43
44	2.96381	3.67145	4.54334	5.61652	6.93612	8.55715	44
45	3.03790	3.78160	4.70236	5.84118	7.24825	8.98501	45
46	3.11385	3.89504	4.86694	6.07482	7.57442	9.43426	46
47	3.19170	4.01190	5.03728	6.31782	7.91527	9.90597	47
48	3.27149	4.13225	5.21359	6.57053	8.27146	10.40127	48
49	3.35328	4.25622	5.39606	6.83335	8.64367	10.92133	49
50	3.43711	4.38391	5.58493	7.10668	9.03264	11.46740	50

TABLE II.

Present Value of 1: viz., v^n.

n	$2\frac{1}{2}\%$	3%	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	n
1	.97561	.97087	.96618	.96154	.95694	.95238	1
2	.95181	.94260	.93351	.92456	.91573	.90703	2
3	.92860	.91514	.90194	.88900	.87630	.86384	3
4	.90595	.88849	.87144	.85480	.83856	.82270	4
5	.88385	.86261	.84197	.82193	.80245	.78353	5
6	.86230	.83748	.81350	.79031	.76790	.74622	6
7	.84127	.81309	.78599	.75992	.73483	.71068	7
8	.82075	.78941	.75941	.73069	.70319	.67684	8
9	.80073	.76642	.73373	.70259	.67290	.64461	9
10	.78120	.74409	.70892	.67556	.64393	.61391	10
11	.76214	.72242	.68495	.64958	.61620	.58468	11
12	.74356	.70138	.66178	.62460	.58966	.55684	12
13	.72542	.68095	.63940	.60057	.56427	.53032	13
14	.70773	.66112	.61778	.57748	.53997	.50507	14
15	.69047	.64186	.59689	.55526	.51672	.48102	15
16	.67362	.62317	.57671	.53391	.49447	.45811	16
17	.65720	.60502	.55720	.51337	.47318	.43630	17
18	.64117	.58739	.53836	.49363	.45280	.41552	18
19	.62553	.57029	.52016	.47464	.43330	.39573	19
20	.61027	.55368	.50257	.45639	.41464	.37689	20
21	.59539	.53755	.48557	.43883	.39679	.35894	21
22	.58086	.52189	.46915	.42196	.37970	.34185	22
23	.56670	.50669	.45329	.40573	.36335	.32557	23
24	.55288	.49193	.43796	.39012	.34770	.31007	24
25	.53939	.47761	.42315	.37512	.33273	.29530	25
26	.52623	.46369	.40884	.36069	.31840	.28124	26
27	.51340	.45019	.39501	.34682	.30469	.26785	27
28	.50088	.43708	.38165	.33348	.29157	.25509	28
29	.48866	.42435	.36875	.32065	.27902	.24295	29
30	.47674	.41199	.35628	.30832	.26700	.23138	30
31	.46511	.39999	.34423	.29646	.25550	.22036	31
32	.45377	.38834	.33259	.28506	.24450	.20987	32
33	.44270	.37703	.32134	.27409	.23397	.19987	33
34	.43191	.36604	.31048	.26355	.22390	.19035	34
35	.42137	.35538	.29998	.25342	.21425	.18129	35
36	.41109	.34503	.28983	.24367	.20503	.17266	36
37	.40107	.33498	.28003	.23430	.19620	.16444	37
38	.39128	.32523	.27056	.22529	.18775	.15661	38
39	.38174	.31575	.26141	.21662	.17967	.14915	39
40	.37243	.30656	.25257	.20829	.17193	.14205	40
41	.36335	.29763	.24403	.20028	.16453	.13528	41
42	.35448	.28896	.23578	.19257	.15744	.12884	42
43	.34584	.28054	.22781	.18517	.15066	.12270	43
44	.33740	.27237	.22010	.17805	.14417	.11686	44
45	.32917	.26444	.21266	.17120	.13796	.11130	45
46	.32115	.25674	.20547	.16461	.13202	.10600	46
47	.31331	.24926	.19852	.15828	.12634	.10095	47
48	.30567	.24200	.19181	.15219	.12090	.09614	48
49	.29822	.23495	.18532	.14634	.11569	.09156	49
50	.29094	.22811	.17905	.14071	.11071	.08720	50

TABLE III.

Amount of 1 per Annum: viz., $s_{\overline{n}|}$.

n	$2\frac{1}{2}\%$	3%	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	n
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1
2	2.0250	2.0300	2.0350	2.0400	2.0450	2.0500	2
3	3.0756	3.0909	3.1062	3.1216	3.1370	3.1525	3
4	4.1525	4.1836	4.2149	4.2465	4.2782	4.3101	4
5	5.2563	5.3091	5.3625	5.4163	5.4707	5.5256	5
6	6.3877	6.4684	6.5502	6.6330	6.7169	6.8019	6
7	7.5474	7.6625	7.7794	7.8983	8.0192	8.1420	7
8	8.7361	8.8923	9.0517	9.2142	9.3800	9.5491	8
9	9.9545	10.1591	10.3685	10.5828	10.8021	11.0266	9
10	11.2034	11.4639	11.7314	12.0061	12.2882	12.5779	10
11	12.4835	12.8078	13.1420	13.4864	13.8412	14.2068	11
12	13.7956	14.1920	14.6020	15.0258	15.4640	15.9171	12
13	15.1404	15.6178	16.1130	16.6268	17.1599	17.7130	13
14	16.5190	17.0863	17.6770	18.2919	18.9321	19.5986	14
15	17.9319	18.5989	19.2957	20.0236	20.7841	21.5786	15
16	19.3802	20.1569	20.9710	21.8245	22.7193	23.6575	16
17	20.8647	21.7616	22.7050	23.6975	24.7417	25.8404	17
18	22.3863	23.4144	24.4997	25.6454	26.8551	28.1324	18
19	23.9460	25.1169	26.3572	27.6712	29.0636	30.5390	19
20	25.5447	26.8704	28.2797	29.7781	31.3714	33.0660	20
21	27.1833	28.6765	30.2695	31.9692	33.7831	35.7193	21
22	28.8629	30.5368	32.3289	34.2480	36.3034	38.5052	22
23	30.5844	32.4529	34.4604	36.6179	38.9370	41.4305	23
24	32.3490	34.4265	36.6665	39.0826	41.6892	44.5020	24
25	34.1578	36.4593	38.9499	41.6459	44.5652	47.7271	25
26	36.0117	38.5530	41.3131	44.3117	47.5706	51.1135	26
27	37.9120	40.7096	43.7591	47.0842	50.7113	54.6631	27
28	39.8598	42.9309	46.2906	49.9676	53.9933	58.4026	28
29	41.8563	45.2189	48.9108	52.9663	57.4230	62.3227	29
30	43.9027	47.5754	51.6227	56.0849	61.0071	66.4388	30
31	46.0003	50.0027	54.4295	59.3283	64.7524	70.7608	31
32	48.1503	52.5028	57.3345	62.7015	68.6662	75.2988	32
33	50.3540	55.0778	60.3412	66.2095	72.7562	80.0638	33
34	52.6129	57.7302	63.4532	69.8579	77.0303	85.0670	34
35	54.9282	60.4621	66.6740	73.6522	81.4966	90.3203	35
36	57.3014	63.2759	70.0076	77.5983	86.1640	95.8363	36
37	59.7339	66.1742	73.4579	81.7022	91.0413	101.6281	37
38	62.2273	69.1594	77.0289	85.9703	96.1382	107.7095	38
39	64.7830	72.2342	80.7249	90.4091	101.4644	114.0950	39
40	67.4026	75.4013	84.5503	95.0255	107.0303	120.7998	40
41	70.0876	78.6633	88.5095	99.8265	112.8467	127.8398	41
42	72.8398	82.0232	92.6074	104.8196	118.9248	135.2318	42
43	75.6608	85.4839	96.8486	110.0124	125.2764	142.9933	43
44	78.5523	89.0484	101.2383	115.4129	131.9138	151.1430	44
45	81.5161	92.7199	105.7817	121.0294	138.8500	159.7002	45
46	84.5540	96.5015	110.4840	126.8706	146.0982	168.6852	46
47	87.6679	100.3965	115.3510	132.9454	153.6726	178.1194	47
48	90.8596	104.4084	120.3883	139.2632	161.5879	188.0254	48
49	94.1311	108.5406	125.6018	145.8337	169.8594	198.4267	49
50	97.4843	112.7969	130.9979	152.6671	178.5030	209.3480	50

TABLE IV.

Present Value of 1 per Annum: viz., $a_{\overline{n}|}$.

n	$2\frac{1}{2}\%$	3%	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	n
1	0.9756	0.9709	0.9662	0.9615	0.9569	0.9524	1
2	1.9274	1.9135	1.8997	1.8861	1.8727	1.8594	2
3	2.8560	2.8286	2.8016	2.7751	2.7490	2.7232	3
4	3.7620	3.7171	3.6731	3.6299	3.5875	3.5460	4
5	4.6458	4.5797	4.5151	4.4518	4.3900	4.3295	5
6	5.5081	5.4172	5.3286	5.2421	5.1579	5.0757	6
7	6.3494	6.2303	6.1145	6.0021	5.8927	5.7864	7
8	7.1701	7.0197	6.8740	6.7327	6.5959	6.4632	8
9	7.9709	7.7861	7.6077	7.4353	7.2688	7.1078	9
10	8.7521	8.5302	8.3166	8.1109	7.9127	7.7217	10
11	9.5142	9.2526	9.0016	8.7605	8.5289	8.3064	11
12	10.2578	9.9540	9.6633	9.3851	9.1186	8.8633	12
13	10.9832	10.6350	10.3027	9.9856	9.6829	9.3936	13
14	11.6909	11.2961	10.9205	10.5631	10.2228	9.8986	14
15	12.3814	11.9379	11.5174	11.1184	10.7395	10.3797	15
16	13.0550	12.5611	12.0941	11.6523	11.2340	10.8378	16
17	13.7122	13.1661	12.6513	12.1657	11.7072	11.2741	17
18	14.3534	13.7535	13.1897	12.6593	12.1600	11.6896	18
19	14.9789	14.3238	13.7098	13.1339	12.5933	12.0853	19
20	15.5892	14.8775	14.2124	13.5903	13.0079	12.4622	20
21	16.1845	15.4150	14.6980	14.0292	13.4047	12.8212	21
22	16.7654	15.9369	15.1671	14.4511	13.7844	13.1630	22
23	17.3321	16.4436	15.6204	14.8568	14.1478	13.4886	23
24	17.8850	16.9355	16.0584	15.2470	14.4955	13.7986	24
25	18.4244	17.4131	16.4815	15.6221	14.8282	14.0939	25
26	18.9506	17.8768	16.8904	15.9828	15.1466	14.3752	26
27	19.4640	18.3270	17.2854	16.3296	15.4513	14.6430	27
28	19.9649	18.7641	17.6670	16.6631	15.7429	14.8981	28
29	20.4535	19.1885	18.0358	16.9837	16.0219	15.1411	29
30	20.9303	19.6004	18.3920	17.2920	16.2889	15.3725	30
31	21.3954	20.0004	18.7363	17.5885	16.5444	15.5928	31
32	21.8492	20.3888	19.0689	17.8736	16.7889	15.8027	32
33	22.2919	20.7658	19.3902	18.1476	17.0229	16.0025	33
34	22.7238	21.1318	19.7007	18.4112	17.2468	16.1929	34
35	23.1452	21.4872	20.0007	18.6646	17.4610	16.3742	35
36	23.5563	21.8323	20.2905	18.9083	17.6660	16.5469	36
37	23.9573	22.1672	20.5705	19.1426	17.8622	16.7113	37
38	24.3486	22.4925	20.8411	19.3679	18.0500	16.8679	38
39	24.7303	22.8082	21.1025	19.5845	18.2297	17.0170	39
40	25.1028	23.1148	21.3551	19.7928	18.4016	17.1591	40
41	25.4661	23.4124	21.5991	19.9931	18.5661	17.2944	41
42	25.8206	23.7014	21.8349	20.1856	18.7235	17.4232	42
43	26.1664	23.9819	22.0627	20.3708	18.8742	17.5459	43
44	26.5038	24.2543	22.2828	20.5488	19.0184	17.6628	44
45	26.8330	24.5187	22.4955	20.7200	19.1563	17.7741	45
46	27.1542	24.7754	22.7009	20.8847	19.2884	17.8801	46
47	27.4675	25.0247	22.8994	21.0429	19.4147	17.9810	47
48	27.7732	25.2667	23.0912	21.1951	19.5356	18.0772	48
49	28.0714	25.5017	23.2766	21.3415	19.6513	18.1687	49
50	28.3623	25.7298	23.4556	21.4822	19.7620	18.2559	50

LECTURE V.

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Annuities-Certain.

An Annuity is a series of payments made at equal intervals during the continuance of a given status.

The status, or condition upon which the payment of the annuity depends, may, of course, be anything agreed upon by the contracting parties, and hence may take a variety of forms. In case the status is a fixed term of years the annuity is called an annuity-certain. Thus a series of payments of five dollars a year for twenty years would constitute an annuity-certain. The sum of the payments made on an annuity in one year (in case the payments are uniform, that is, of the same amount) is called the annual rent.

A agrees to pay B three dollars at the end of every month for ten years. This is an annuity-certain with annual rent of thirty-six dollars.

When the payments are made at the end of the interval the annuity-certain is said to be immediate, when the payments are made at the beginning of the interval the annuity-certain is said to be due.

Thus the above annuity is an immediate annuity-certain with twelve payments each year.

A agrees to pay B two dollars every three months for seven years the first payment to be made at once.

This is a due annuity-certain (or, as it is often called, an annuity due) with four payments per year and an annual rent of eight dollars.

We shall confine ourselves for the present to immediate annuities-certain with one payment each year. Consider graphically the immediate annuity-certain for n years with annual rent 1, that is, a payment of 1 at the end of each year for n years.

On a line we lay off to the right from a point O a unit space to represent one year, two spaces for two years, etc. We thus have a clear graphic picture of the payments of the annuity as they occur.

```
Payments        1        1        1                       1        1
--|-------------+--------+--------+-----------------------+--------+--
   1 yr     1 yr     1 yr                             1 yr
            1 yr     2 yrs    3 yrs                   n-1 yrs  n yrs
```

Our first problem will be to determine the present value of the above annuity. Designating the present value by $a_{\overline{n}|}$, we see that it is equal to the present value of 1 due one year hence plus the present value of 1 due two years hence, etc...... plus the present value of 1 due n years hence. By formula (10)

the present value of 1 due in 1 year at rate of interest i is v

" " " " 1 " " 2 years " " " " i " v^2

....

" " " " 1 " " n " " " " " i " v^n

Hence adding we have

$$a_{\overline{n}|} = v + v^2 + v^3 + \dots\dots\dots + v^n$$
$$= v(1 + v + v^2 + v^3 + \dots\dots\dots + v^{n-1})$$
$$= v(1 - v^n) / (1 - v) = (1 - v^n) / (1/v - 1)$$
$$= (1 - v^n) / (1 + i - 1) = (1 - v^n) / i.$$

We have finally then

$$a_{\overline{n}|} = (1 - v^n) / i \qquad (24)$$

Referring back to formula (13) we see that the numerator of (24) is the total discount on 1 due in n years at effective rate of interest i while the denominator is the effective rate of interest, i. Hence we may write for this particular form of annuity-certain

$$a_{\overline{n}|} = \frac{\text{The total discount on 1 due in n years}}{\text{The effective rate of interest}} \qquad (25)$$

Example.--What is the present value at 3% of an annual payment of \$375 at the end of each year for ten years?

Solution.--The present value of a similar payment of 1 per annum is $a_{\overline{10}|}$, hence we have

Present value = 375 $a_{\overline{10}|}$.

Referring to the table we find that $a_{\overline{10}|}$ @ 3% is 8.5302 and therefore the required present value is 375 x 8.5302 = 3198.82. The present value of the series of payments is then \$3198.82.

In arriving at formula (24) we were obliged to sum a geometric series and as this form of series frequently comes up in actuarial work it will be worth while to devote a few words to the method of summing such series at this point. The general form of a geometric series is the following:

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6 + \ldots\ldots + ar^{n-1} + ar^n + \ldots$$

The first term is a, the <u>ratio</u> is r; let it be required to find the sum of the first n terms of this series. It will be noticed that the first term does not involve r, the second involves r to

the first power, the third involves r to the second power,........, the nth term involves r to the power n-1. Designating the sum of the first n terms by s we have

$$s = a + ar + ar^2 + ar^3 + ar^4 + \dots\dots\dots + ar^{n-1}. \tag{26}$$

If we now multiply both sides of (26) by r and subtract the same from formula (26) we shall be led to the following scheme:

$$\begin{array}{rl} s &= a + ar + ar^2 + ar^3 + ar^4 + \dots\dots + ar^{n-2} + ar^{n-1} \\ sr &= \quad\;\; ar + ar^2 + ar^3 + ar^4 + \dots\dots + ar^{n-2} + ar^{n-1} + ar^n \\ \hline s - sr &= a \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - ar^n \end{array}$$

This may be written

$s(1 - r) = a(1 - r^n)$ and dividing both sides by $(1 - r)$ we are led to the desired formula expressing the sum of the first n terms in terms of a, r and n:

$$s = a\,\frac{1 - r^n}{1 - r} \tag{27}$$

Let us apply (27) to sum the series on page 25, that is,

$$s = v + v^2 + v^3 + \dots\dots\dots + v^n.$$

Here we have a = v, r = v, and the number of terms is n. Hence substituting in (27) it appears that

$$s = v\,\frac{1 - v^n}{1 - v} = a_{\overline{n}|}.$$

A geometric series of frequent occurrence is the following:

$$1 + x + x^2 + x^3 + x^4 + \dots\dots\dots + x^{n-1}. \tag{28}$$

To sum this we note that a = 1, r = x, and n = n. Substituting in (27) we have

$$s = 1\,\frac{1 - x^n}{1 - x} = (1 - x^n) \;/\; (1 - x) \tag{29}$$

Formula (29) may be easily verified by ordinary long division

as below:

$$
\begin{array}{l}
1 - x \,)\ \ 1 - x^n \ \ (\ 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots\ldots + x^{n-1} \\
\qquad\quad \underline{1 - x} \\
\qquad\qquad\quad x - x^n \\
\qquad\qquad\quad \underline{x - x^2} \\
\qquad\qquad\qquad\quad x^2 - x^n \\
\qquad\qquad\qquad\quad \underline{x^2 - x^3} \\
\qquad\qquad\qquad\qquad\quad x^3 - x^n \\
\qquad\qquad\qquad\qquad\qquad\quad \ldots\ldots \\
\qquad\qquad\qquad\qquad\qquad\qquad \ldots\ldots \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad x^{n-1} - x^n \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x^{n-1} - x^n} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

In a similar manner the truth of formula (27) may be verified by long division; * this is left as an exercise for the student.

*1. A deposit of \$146.75 is to be made in a bank at the end of every year for 27 years. The deposits are to be compounded annually at 4%; what is the present value of these deposits?

*2. Find the formula giving the relation between an annuity due for n years and an annuity immediate for n years, both at the same rate of interest. The annuity due for n years is denoted by the symbol $a_{\overline{n}|}$.

*3. Make up and solve two examples illustrating theory developed in this lesson.

*4. A gentleman pays down $1004 for an immediate annuity of $100 per annum; how long must he live to realize 3% compound interest on the investment? Four per cent?

Suggestion.--Solve (24) for n by logarithms, noting that $v = 1 / (1 + i)$.

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LECTURE VI.

In the preceding lesson we obtained formula (24) involving three quantities, namely: i, n, and $a_{\overline{n}|}$. It is to be noted that v, which appears therein, is really a function of i, for $v = 1 / (1 + i)$.

We may express the formula thus:

$$a_{\overline{n}|} = \frac{1 - \dfrac{1}{(1+i)^n}}{i} \qquad (30)$$

or, using negative exponents,

$$a_{\overline{n}|} = \frac{1 - (1+i)^{-n}}{i} \qquad (31)$$

Now if any two of the three quantities i, n, $a_{\overline{n}|}$, are given, theoretically the third can be obtained. Formula (25), for example, expresses $a_{\overline{n}|}$ in terms of i and n. Our next problem will be to express n in terms of i and $a_{\overline{n}|}$. From (30) we have

$$ia_{\overline{n}|} = 1 - 1 / (1+i)^n$$

hence $$1 / (1+i)^n = 1 - ia_{\overline{n}|}$$

and $$(1+i)^n = 1 / (1 - ia_{\overline{n}|})$$

Taking the logarithm of each side of this equation we have

$$\log(1+i)^n = \log(1 / (1 - ia_{\overline{n}|}))$$
$$= \log 1 - \log(1 - ia_{\overline{n}|})$$

But since the logarithm of a power is equal to the power times the logarithm of the base, and since $\log 1 = 0$, this becomes,

$$n \log(1 + i) = -\log(1 - ia_{\overline{n}|})$$

hence
$$n = -\frac{\log(1 - ia_{\overline{n}|})}{\log(1 + i)} \qquad (32)$$

or
$$n = \frac{\log(1 - ia_{\overline{n}|})}{\log v} \qquad (32a)$$

*Problem.--$2396.37 is paid for an immediate annuity-certain of $146.75; how many years has it to run if interest is 4%?

The student should note that owing to the negative characteristic the numerator, $\log(1 - ia_{\overline{n}|})$, in (32) will always be negative, and that taken with the minus sign makes the fraction positive, since the $\log(1 + i)$ is always positive. For example, the log.6988 is $\bar{1}.844353 = -1 + .844353 = -.155647$. The student should also remember that the annual rent or payment of the annuity $a_{\overline{n}|}$ is 1.

The case when $a_{\overline{n}|}$ and n are given to find the rate of interest i involves the solution of an equation of higher degree and would not properly come under an elementary treatment such as is proposed in this course. If a table of the values of $a_{\overline{n}|}$ for various rates of interest, however, is available, an approximate solution of this case may be obtained by simple proportion. See problem at end of lesson.

We next proceed to find the amount of an immediate annuity-certain with an annual payment of 1.

To find the amount we must accumulate at the given rate of interest all the annual payments to the point E, in time n years ahead. For figure see page 34. We thus have to accumulate the

first payment of 1 for n - 1 years, the second payment of 1 for n - 2 years, etc., ... the (n - 1)th payment of 1 for 1 year, the nth payment of 1 for 0 years, at compound interest at the rate i. By formula (9) the amount of 1 for n years at rate i is $(1 + i)^n$. Hence we have:

Amount	of	1	at	rate	i	for	n - 1	years	is	$(1 + i)^{n-1}$
"	"	1	"	"	i	"	n - 2	"	"	$(1 + i)^{n-2}$
......										
"	"	1	"	"	i	"	2	"	"	$(1 + i)^2$
"	"	1	"	"	i	"	1	"	"	$(1 + i)$
"	"	1	"	"	i	"	0	"	"	1

Adding the separate amounts we have, denoting the total amount by the symbol $s_{\overline{n}|}$,

$$s_{\overline{n}|} = 1 + (1 + i) + (1 + i)^2 + (1 + i)^3 + \ldots\ldots + (1 + i)^{n-1}.$$

This is a geometric series whose first term is 1, ratio (1 + i) and number of terms n. Hence by formula (27) we have

$$s_{\overline{n}|} = 1.\ \frac{1 - (1 + i)^n}{1 - (1 + i)} = \frac{(1 + i)^n - 1}{i} \qquad (33)$$

This is a fundamental formula and is of much importance in the theory of compound interest. Referring to formula (12) we see that the numerator of (33), $(1 + i)^n - 1$, is the total interest on one unit for n years and this leads to the verbal equation:

$$s_{\overline{n}|} = \frac{\text{Total interest on 1 for n years}}{\text{The effective rate of interest}} \qquad (34)$$

The student should note the analogy between formulas (25) and (34), and also (31) and (33).

Formula (33) involves three quantities: i, n, $s_{\overline{n}|}$, the latter

being expressed in terms of i and n. By solving for n we find that

$$n = \frac{\log(1 + is_{\overline{n|}})}{\log(1 + i)} \qquad (35)$$

*The student should work this out. This formula with the aid of logarithms enables us to determine the number of years or interval of time when the accumulation or amount and the rate of interest are given.

Tables of $a_{\overline{n|}}$ and $s_{\overline{n|}}$ have been computed and in working problems involving these quantities it is of course needless for the student to compute them. In the "Short Collection of Actuarial Tables" $a_{\overline{n|}}$ and $s_{\overline{n|}}$ are given to four places of decimals and $1 / a_{\overline{n|}}$ or, as it may otherwise be written, $a_{\overline{n|}}^{-1}$, to six places of decimals. They are all given for values of n from 1 to 50 and at rates of interest varying by 1/2% from $2\frac{1}{2}$% to 5% inclusive.

Example.--What is the amount at 3% of an annual payment $375 at end of each year for ten years?

The amount of a similar payment of 1 for ten years is by the $s_{\overline{n|}}$ table: $s_{\overline{10|}}$ @ 3% = 11.4639. Hence we have $375s_{\overline{10|}}$ = 375 x 11.4639 = 4298.96. The total accumulation is therefore $4298.96.

Problems.

*1. Beginning January 1 a deposit of $25 is made every six months in a savings bank where interest is 5% compounded semi-annually. What will be the accumulation at the time the 42d deposit is made?

*2 Prove that $1/a_{\overline{n|}} = i + 1/s_{\overline{n|}}$

*3. Compute $a_{\overline{42}|}$ and $s_{\overline{42}|}$ when $i = 1\frac{1}{2}\%$. Find to four decimal places using logarithms. Then solve example 1 when interest is 3% compounded semi-annually.

*4. An immediate annuity-certain of \$125 per annum at 4% amounts after how many years to \$10746.29.

*5. Show how by proportion with the aid of a table of $a_{\overline{n}|}$ and $s_{\overline{n}|}$ the rate of interest, i, on an annuity-certain may be approximately obtained if the quantities, $a_{\overline{n}|}$, n, or $s_{\overline{n}|}$, n, are given.

*6. A gentleman at age 35 takes a policy with annual premium of \$27.63; if he lives to make his 17th payment on this policy, what will be the amount of all the premiums he has made the company at that time assuming money to be worth three and one-half per cent?

*7. In the preceding problem how many years must he live in order to have made premium payments in excess of the face, \$1000, of the policy? (a) if money is worth 3%, (b) if money is worth 4%.

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L E S S O N VII.

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APPLICATION OF THE PRECEDING LESSONS TO MISCELLANEOUS PROBLEMS.

One of the most important applications of the Theory of Compound Interest and Annuities is to the valuation of securities. We shall first determine how to find the value of a loan, repayable by instalments at stated periods of time, with interest (or dividend) on the outstanding portion of the loan at a rate g, and all computed, or <u>valued</u>, so as to yield the purchaser a given rate of interest i. A good illustration of this is furnished by a Municipal Bond. In the Commercial & Financial Chronicle of Nov. 30, 1901, p. 1175, the following Bond proposal is made.

> "Huntsville, Alabama, Bond Offering.--Proposals will be received until Jan. 7, 1902, for an issue of $40,000, 5% Gold School Bonds. Securities are in denomination of $500, dated Jan. 1, 1902. Interest will be payable in New York City. Principal will mature Jan. 1, 1932."

Here we have a loan of $40,000 made by the purchaser of the securities or bonds. The annual interest (or dividend) on these bonds is 5%. The bonds are redeemed at the end of 30 years. Suppose an intending purchaser of these bonds wished to pay a price so as to yield him a net rate of 4%, how much ought he to pay? This is the nature of the general problem we are to discuss. It is to be noted that if the purchaser of this loan to the City of Huntsville

wished to realize 5% on his investment, he would pay \$40,000 for the bonds, or \$1 for each dollar to be redeemed. If, however, he is content with 4%, more than \$40,000 must be paid for the bonds,-- i. e., more than \$1 for each dollar to be redeemed. Let us now proceed to the general case.

Let C represent the price to be paid on redemption.

Let n represent the number of years after which the security becomes redeemable.

Let K represent the present value of C, due n years hence, at the rate of interest i employed in the valuation of the security.

Let g represent the ratio of the dividend per annum to C.

Let A represent the present value of the security.

In the above example, $C = 40000$, $n = 30$. The dividend or interest per annum is $.05 \times 40000 = 2000$. Hence $g = 2000 / 40000 = .05$. By Formula (10) the present value of 40000 due in 30 yrs. at 4% is $v^{30} \times 40000$, where v is $\frac{1}{1.04}$, i. e., the value of v corresponding to $i = 4\%$.

Now, returning to the general problem, we note that the value of the security, so far as the purchaser or holder is concerned, consists of two parts:--1st, the annual interest (or dividend) to be received; and 2d, the sum to be redeemed at the end of n yrs. To find the present value, A, of the security, then, we must find the value of each of these two parts and add them together. How much is the annual dividend or interest? The interest per unit the redemption price C is by definition g; if the interest on it is g, the interest on C units is gC. Hence at the end of every year for n yrs. the holder will receive gC units.

$$\begin{array}{ccccccc} & gC & gC & gC - - - - - - - - - & gC & gC & \text{and } C \\ \bullet & 1\text{ yr} & 2\text{ yrs} & 3\text{ yrs} - - - - - & (n-1)\text{ yrs} & n\text{ yrs} & \end{array}$$

Now, it is evident that these interest or dividend payments of gC at the end of every year constitute an immediate annuity-certain of annual rent gC and period of n yrs. The value of such an annuity with annual rent 1 is $a_{\overline{n}|}$; hence the value of the annuity with annual rent gC is

$$g\,C\,a_{\overline{n}|}$$

It is to be noted here that $a_{\overline{n}|}$ is to be taken at the rate of interest i to be employed in the valuation of the security, a rate which in general is different from g, the rate of dividend.

The present value of the sum C, to be redeemed in n yrs., is by (10), $v^n \times C = K$, by definition.

Adding these results together we have, $A = v^nC + gCa_{\overline{n}|}$

By (24) $a_{\overline{n}|} = \dfrac{1 - v^n}{i}$. Substituting in the above formula we have

$$A = v^nC + gC\left[\frac{1 - v^n}{i}\right]$$

$$= v^nC + \frac{g}{i}(C - v^nC)$$

Since $K = v^nC$, $\therefore$ $A = K + \frac{g}{i}(C - K)$

If in the above formula we regard the total sum to be redeemed as unity, then C = 1, and K is the present value of 1 to be redeemed in n yrs., i. e., $K = v^nC = v^n1 = v^n$, and we have,

$$A = K + (g/i)(1-K) \tag{36}$$

or, $A = v^n + (g/i)(1 - v^n)$ adding and subtracting 1, we have, $A = 1 - (1 - v^n) + (g/i)(1-v^n)$

therefore $A = 1 - (1 - v^n)(1 - g/i)$

$$= 1 - \left[\frac{1 - v^n}{i}\right](i - g), \text{ but } \frac{1 - v^n}{i} = a_{\overline{n}|}$$

$$\therefore\ A = 1 - a_{\overline{n}|}(i - g) \tag{37}$$

$A = 1 + a_{\overline{n}|}(g - i)$

In this formula $a_{\overline{n}|}$ is computed at 1%.

Formula (37) thus gives the value of a security where the sum to be redeemed is 1. Knowing this, of course, we can find the value of a security of the same nature where the sum to be redeemed is any given amount. A, then, is the purchase price per unit of the redemption price. Now, A may be greater than 1, in which case the security is said to be purchased at a <u>premium</u>; in case A is less than 1, the security is purchased at a <u>discount</u>. If we subtract 1 from both sides of (37) we have:

$$A - 1 = -a_{\overline{n}|}^{1\%}(i - g) = a_{\overline{n}|}^{1\%}(g - i)$$

Now, let the excess of A over 1 be denoted by k; then,

$$k = a_{\overline{n}|}^{1\%}(g - i) \qquad (38)$$

We see thus that if k is positive, the security is at a premium; and if k is negative it is at a discount. Since $a_{\overline{n}|}$ is always positive, it appears from (38) that the sign of k will be positive when $(g - i)$ is positive, i. e., when g is greater than i,--or <u>the rate of dividend is greater than the rate of interest used in valuation</u>; conversely, when g is less than i, we shall have k negative.

Let us now use our results to find the value of the Huntsville, Ala., security, on the hypothesis that the purchaser wishes to realize 4% on his investment. Consider a dollar (unit) of the loan C = 40000.

Here $n = 30$. $g = .05$. $i = .04$.

By (38) $k = a_{\overline{30}|}^{4\%}(.05 - .04)$ $\quad k = a_{\overline{n}|}(g - i)$

$= 17.2920 \times .01 = .17292$

or the premium is slightly over 17¢ on the dollar. Since for each dollar of the loan the purchaser must pay \$1.17292, for the whole loan of \$40000 he must pay

$$40000 \times 1.17292 = 46916.80.$$

Hence the bid on the total loan, in order to realize 4% compound interest on the investment, is \$46,916.80.

1. Find the bid when the rates to be realized are: (a) $4\frac{1}{2}$%, (b) $3\frac{1}{2}$%, (c) 3%, (d) $2\frac{1}{2}$%.

2. Purchase the issue of the Financial & Commercial Chronicle dated June 6, 1903, #1980, and turn to page 1263 and value the Harrodsburg, Ky., Bond proposal at $3\frac{1}{2}$%. It is as follows: \$18000 Bonds (coupon), due in 20 yrs. bearing interest at 4%, payable January and July of each year. (You may assume that interest is payable annually in January of each year.)

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L E S S O N VIII.

VALUATION OF SECURITIES (Continued).

In the preceding lesson it was shown that (38)

$$k = a_{\overline{n|}}(g - i)$$

where k is the premium paid for each unit of the loan to be redeemed after n years.

In case the interest is nominal, say j/m, and the dividend payments per unit of the loan to be redeemed are paid in m equal instalments, g/m, during the year, it is evident that we have a case of n x m intervals with g/m as dividend and j/m as interest. Hence the above formula becomes,--

$$k = a_{\overline{nm|}}^{j/m\%}\left[\frac{g - j}{m}\right] \tag{39}$$

In particular, if interest is $j/2$, and dividend payment semi-annual,

$$k = a_{\overline{2n|}}^{j/2\%}\left[\frac{g - j}{2}\right] \tag{40}$$

We shall now pass to the more general case of the valuation of a security where it is not redeemed in one sum, but in a series of payments. First consider the simpler case where the dividend payments are annual and the rate of interest is the effective rate i.

Let C_1, C_2 .....C_r represent the successive instalments by which the principal is to be redeemed.

" n_1, n_2 .....n_r " the respective number of years after which the successive instalments become due.

" K_1, K_2 .....K_r " the present values, at the valuation rate of interest i, of

C_1 due n_1 years hence

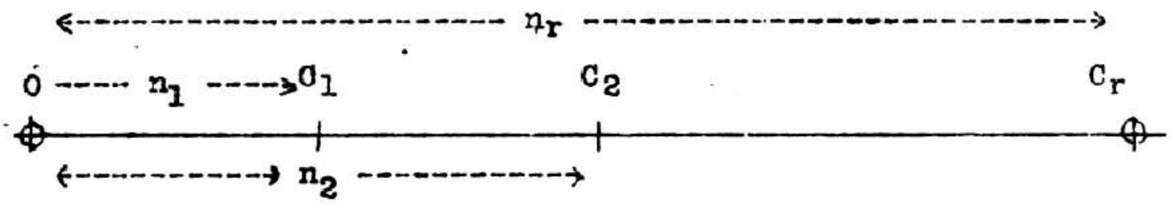
C_2 due n_2 " "

to

C_r due n_r " "

" g " the fixed rate of dividend to be paid on the outstanding investments.

" A_1, A_2 .....A_r " the present values, at the valuation rate i, of the separate instalments with their respective dividends.



Now, each payment may be regarded as a distinct problem under the head of the preceding lesson, so in order to value the total security we may regard it as made up of r different loans, and after finding the value of each one add them all together for the value of the total security.

We saw that in the case of a single loan of C_1, at valuation rate i, dividend rate g, due in n_1 years, the present value A_1, of the security is:

$$A_1 = K_1 + (g/i)(C_1 - K_1)$$

Similarly $A_2 = K_2 + (g/i)(C_2 - K_2)$

$$A_3 = K_3 + (g/i)(C_3 - K_3)$$

to

$$\underline{A_r = K_r + (g/i)(C_r - K_r)}$$

Adding $(A_1 + A_2 + \ldots + A_r) = (K_1 + K_2 + \ldots + K_r) + (g/i)\left[(C_1 + C_2 + \ldots + C_r) - (K_1 + K_2 + \ldots + K_r)\right]$

But if we denote the total sum to be redeemed, i. e., $C_1 + C_2 + \ldots + C_r$, by C; the total present worth of C_1 in n_1 years, C_2 in n_2 years, etc., i. e., $K_1 + K_2 + \ldots + K_r$, by K; and the total value of the security, the sum of the several parts $A_1 + A_2 + \ldots + A_r$ by A; we have;--

$$A = K + (g/i)(C - K) \tag{41}$$

This may be put in another convenient form by a little algebraic transformation. Add and subtract C from right hand member of (41), and we have:--

$$A = C - C + K + (g/i)(C - K)$$

putting (-C + K) in parentheses preceded by a negative sign, for $(-C + K) = -(C - K)$:

$$A = C - (C - K) + (g/i)(C - K)$$

Bringing first term to left hand member:

$$A - C = -(C - K) + (g/i)(C - K)$$

Changing signs in both members:

$$C - A = (C - K) - (g/i)(C - K)$$

Factoring second member:

$$C - A = (C - K)(1 - g/i)$$

Reducing second factor to common denominator i:

$$C - A = (C - K)\left[\frac{i - g}{i}\right]$$

Changing signs of both members:

$$A - C = (C - K)\left[\frac{g - i}{i}\right]$$

If, now, we consider a unit of the total sum to be redeemed, then the above formula gives, when C = 1,

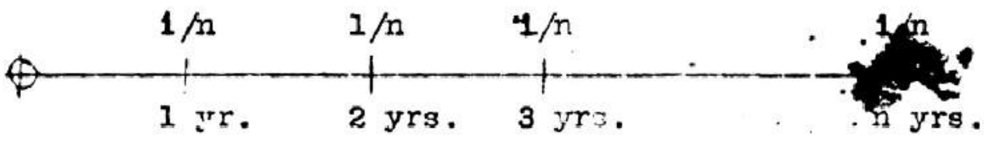
$$A - 1 = (1 - K)\left[\frac{g - i}{i}\right] \qquad (42)$$

where A is the value of each unit of the sum to be redeemed, and K is the present value of the various parts of the unit (at rate i), due in n_1, n_2.... n_r years. Letting A - 1 = k, we have:

$$k = (1 - K)\left[\frac{g - i}{i}\right] \qquad (43)$$

Notice that k, the premium, is positive if g is greater than i, and negative (a discount) if g is less than i; for the first factor (1 - K) can not be negative, as K by definition is the present value of a series of future payments which sum to 1, and hence their present discounted value must be less than 1. This shows us that a security must be purchased at a premium, if it is valued at a lower rate i than the rate of dividend g; and at a discount, if it is valued at a higher rate i than the rate of dividend g.

Let us now apply the general formula (43) to the case of a security redeemed by n equal annual instalments. Consider a unit of the total sum to be redeemed. Since this unit is to be redeemed in n equal payments over n years, the annual portion redeemed is 1/n.



Now, the present value, K, of these n payments is clearly the value of an annuity of 1/n annual rent.

$$\therefore \quad K = a_{\overline{n}|} \times 1/n = \frac{a_{\overline{n}|}}{n}$$

Substituting this value of K in (43), we have:

$$k = \left[1 - \frac{a_{\overline{n}|}}{n}\right]\left[\frac{g - i}{i}\right] \tag{44}$$

*Example 1.--Find the value of the following bonds at 5%, $4\frac{1}{2}$%, 4%, $3\frac{1}{2}$%, 3%, and $2\frac{1}{2}$%. Monterey School District, Monterey County, California, $20,000, 5% gold refunding bonds, denomination $1000, dated August 1, 1901. Interest payable annually at Salinas City. Principal will mature one bond yearly on August 1st from 1902 to 1921 inclusive. Financial Chronicle, Vol. 73, p. 150. These are called "serial" bonds.

*Example 2.--These bonds were purchased by E. H. Rollins & Sons, San Francisco, at 104.63, i. e., $1.0463 for each $1 of the bond. Chronicle, Vol. 73, p. 305. Find approximately the rate of interest i at which E. H. Rollins & Sons made their valuation.

*Example 3.--Rock Spring School District, Placer Co., Cal. Bond offering.--Proposals will be received until 2 P. M. June 20th, by H. E. Albee, County Treasurer, for $1800, 6%, 1 - 6 year (serial) bonds. Denomination, $300. Certified check for 10 per cent required. Find value of Rock Spring Bonds, Chronicle of June 6, 1903, when interest is assumed annual. Evaluate for 5%, 4%, 3%.

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L E S S O N IX.

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Valuation of Securities (continued).

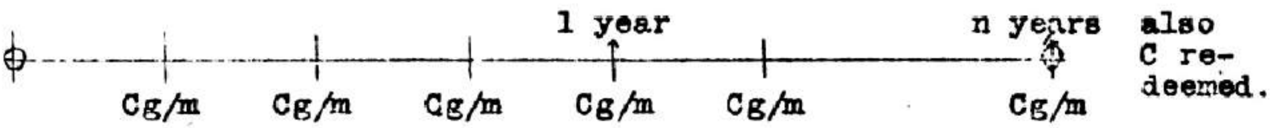
Bond valuation when the annual rate of dividend is g, payable in m instalments each equal to g/m, and the valuation is at a nominal rate of interest j with frequency of conversion m.

Let C be the amount to be redeemed after n years.

Let g be the annual rate of dividend, payable in m instalments of g/m, per unit of the redemption fund C.

Let j be the nominal rate of interest, with frequency of conversion m, to be used in the valuation.

Let A be the value of the security.



Referring to the figure, we see the security consists of: (a) a series of dividend payments of Cg/m each every mth part of a year for n years, plus (b) the final sum C to be redeemed at end of n years. The first part (a) may be considered as an immediate annuity-certain with payments at intervals of 1/m of a year, each of Cg/m. Since interest is at the rate j/m per interval, we have here the present value of these payments equal to $(Cg/m) \times a_{\overline{nm|}}$ at j/m%, since there are nm payments in the n years.

But from (24) we have,

$$(Cg/m) \times a_{\overline{nm|}}^{j/m\%} = (Cg/m) \times \frac{1 - v^{nm}}{j/m},$$

where v is equal to $\frac{1}{1 + j/m}$. Also since interest is compounded every mth of a year, the present value of C, due in n years, is, $C \times v^{nm}$, v at $j/m\%$. Adding these two portions, we have the present value of the security, namely,

$$A = (Cg/m) \times \frac{1 - v^{nm}}{j/m} + Cv^{nm}.$$

$$\therefore \quad A = Cv^{mn} + (g/j)(C - Cv^{mn}) \tag{45}$$

This formula may be put in another form, as follows. Add and subtract C from the second member,

$$A = C - C + Cv^{mn} + (g/j)(C - Cv^{mn})$$

and enclosing second and third terms in brackets preceded by a negative sign, we have,

$$A = C - (C - Cv^{mn}) + (g/j)(C - Cv^{mn})$$

Now, in the last form the factor $(C - Cv^{mn})$ occurs in the second and third terms of the right hand member. Factoring these terms we have:

$$\begin{aligned} A &= C + (C - Cv^{mn})(-1 + g/j) \\ &= C + (C - Cv^{mn})(g/j - 1) \\ &= C + (C - Cv^{mn})\left[\frac{g - j}{j}\right] \\ &= C + C(1 - v^{mn})\left[\frac{g - j}{j}\right] \\ &= C + C\left[\frac{1 - v^{mn}}{j}\right](g - j), \end{aligned}$$

multiplying the denominator of the second factor by 1/m, and the denominator of the third factor by m, thus leaving the whole unchanged in value (since $1/m \times m = 1$), we have,

$$A = C + C\left[\frac{1 - v^{mn}}{j/m}\right]\left[\frac{g - j}{m}\right]$$

But the second factor, $\frac{1 - v^{mn}}{j/m}$ equals $a_{\overline{mn}|}^{j/m\%}$ hence substituting this value,

$$A = C + C \times a_{\overline{mn}|}^{j/m\%}\left[\frac{g - j}{m}\right] \tag{46}$$

If we set $C = 1$, we shall have the present value of 1 unit of the sum to be redeemed:

$$A = 1 + a_{\overline{mn}|}^{j/m\%}\left[\frac{g - j}{m}\right] \tag{47}$$

$$\therefore \quad k = A - 1 = a_{\overline{mn}|}^{j/m\%}\left[\frac{g - j}{m}\right] \tag{48}$$

a formula obtained before by other means.

We are now prepared to consider the more general problem of valuing a security of the following nature, viz.:

(a) The security is redeemed in r equal instalments.

(b) The first redemption payment is made at the end of f years.

(c) The remaining $r - 1$ redemption instalments are made at intervals of t years.

(d) The annual rate of dividend g is paid in m equal instalments.

(e) The security is valued at the nominal rate $j_{(m)}$.

We first proceed to find the present value A of a security of the above type whose total redemption fund is 1. This known, the value of a similar security with total fund C is found by multiplying the above mentioned value of A by C. Since the unit fund is redeemed in r equal instalments, each one will be $1/r$.

```
                                                                rth and
                            1st pyt.  2d pyt    3d pyt          last pyt.
                              1/r      1/r       1/r               1/r
 Redemption payments
                               ↑        ↑         ↑                 ↑
 ⊕──────────────────────────────────────────────────────────────────⊕
 ←-------f years-------→  |←-t yrs-→|←-t yrs-→
                          |←-----------(r - 1)t years-------------→|

|←------------------------f + t(r - 1) years----------------------→|
                     = total period of the loan.
```

The following is an example of such a security: Baldwin Township School District Bonds, \$60,000, 4%, coupons, interest semi-annual,

dated May 1, 1903, maturity \$10,000 on May 1, 1908, 1913, 1918, 1923, 1928, 1933. Commercial & Financial Chronicle, Vol. 76, p. 716.

```
          1/6          1/6          1/6          1/6          1/6          1/6
|-----------|------------|------------|------------|------------|------------|
| f = 5 yrs | t = 5 yrs  | t = 5 yrs  | t = 5 yrs  | t = 5 yrs  | t = 5 yrs  |
            |<-------------(6 - 1)5 = 25 yrs---------------------------->|
|<------------------5 + 5(6 - 1) = 30 yrs------------------------------->|
                     = total period of the loan.
```

This figure is for 1 unit of the above loan. Here $f = 5$, $t = 5$, $r = 6$, $g = .04$, $m = 2$, $C = 60{,}000$. In this example f happens to equal t, but that, in general, will not be true of course. We are now going to prove that the formula for the value of k, the premium per unit of the total sum to be redeemed, is:

$$k = \left[1 - \frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{r a_{\overline{mt}|}}\right] \left[\frac{g - j}{j}\right] \quad \text{at } j/m\% \qquad (49)$$

that is, the annuity present-values in this formula must be computed at the rate of interest j/m. The most common case in practice is where the dividends are paid semi-annually. Here $m = 2$, and formula (49) becomes:

$$k = \left[1 - \frac{a_{\overline{2(f+tr)}|} - a_{\overline{2f}|}}{r a_{\overline{2t}|}}\right] \left[\frac{g - j}{j}\right] \text{ at } j/2\% \qquad (50)$$

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L E S S O N X.

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Valuation of Securities (continued).

Proof of (49).--Referring to the figure at the bottom of page 56, we see that the total unit loan may be regarded as made up of r loans each of 1/r. To find the present value of the total loan we need only to find the present values of these several partial loans and add them together.

We have the following scheme:

No. of loan	Present value	Due or matures in years	Amount maturing per unit of total sum to be redeemed
1	A_1	$n_1 = f$	$C_1 = 1/r$
2	A_2	$n_2 = f + t$	$C_2 = 1/r$
3	A_3	$n_3 = f + 2t$	$C_3 = 1/r$
		&c., to	
r	A_r	$n_r = f + (r - 1)t$	$C_r = 1/r$

Hence, employing formula (45) for each one of these loans, we have the present values as follows:

$$A_1 = \frac{1}{r}\, v^{mf} + (g/j)\left[\frac{1}{r} - \frac{1}{r}\, v^{mf}\right]$$

$$A_2 = \frac{1}{r}\, v^{m(f+t)} + (g/j)\left[\frac{1}{r} - \frac{1}{r}\, v^{m(f+t)}\right] \qquad \text{all}$$

$$A_3 = \frac{1}{r}\, v^{m(f+2t)} + (g/j)\left[\frac{1}{r} - \frac{1}{r}\, v^{m(f+2t)}\right] \qquad \text{at}$$

&c., to $\qquad j/m\%$

$$A_r = \frac{1}{r}\, v^{m(f+\overline{r-1}t)} + (g/j)\left[\frac{1}{r} - \frac{1}{r}\, v^{m(f+\overline{r-1}t)}\right]$$

Adding left hand column we have A, the present value of 1 unit of total sum to be redeemed. Adding the first column of right hand columns and noting the common factors 1/r and v^{mf} (since, by algebra, $v^{m(f+at)} = v^{mf+mat} = v^{mf} \times v^{mat}$), we have,

$$1/r \; v^{mf} \; (1 + v^{mt} + v^{2mt} + v^{3mt} + \ldots\ldots + v^{(r-1)mt}) \qquad \text{(a)}$$

The bracketed terms in the right hand member have the common factor g/j, the first terms in the bracket add up to 1, while the second terms in the bracket are the same as those added above; hence we have:

$$g/j \left[1 - 1/r \; v^{mf} \; (1 + v^{mt} + v^{2mt} + v^{3mt} + \ldots\ldots + v^{(r-1)mt})\right] \qquad \text{(b)}$$

Now, the portion in the bracket is a geometric series whose first term is 1, ratio v^{mt}, number of terms r. [By algebra $v^{amt} = (v^{mt})^a$, hence the bracket in (a) may be written:

$$1 + v^{mt} + (v^{mt})^2 + (v^{mt})^3 + \ldots\ldots + (v^{mt})^{r-1}.\Big]$$

Therefore, by our formula for summing a geometric series:

$$\text{Sum} = 1 \times \frac{1 - (v^{mt})^r}{1 - v^{mt}} = \frac{1 - v^{mtr}}{1 - v^{mt}}.$$

Adding parts (a) and (b) together, we have:

$$A = \frac{1}{r} v^{mf} \frac{1 - v^{mtr}}{1 - v^{mt}} + (g/j)\left[1 - \frac{1}{r} v^{mf} \frac{1 - v^{mtr}}{1 - v^{mt}}\right]$$

If, for the moment, we set $\frac{1}{r} v^{mf} \frac{1 - v^{mtr}}{1 - v^{mt}} = q$, the preceding formula becomes:

$$\begin{aligned}
A &= q + (g/j)(1 - q). && \text{Adding and subtracting 1,}\\
&= 1 - 1 + q + (g/j)(1 - q). && \text{Putting in parenthesis,}\\
&= 1 - (1 - q) + (g/j)(1 - q). && \text{Factoring out } (1 - q),\\
&= 1 + (1 - q)\,(-1 + g/j) &&\\
&= 1 + (1 - q)\,(g/j - 1) = 1 + (1 - q)\left[\frac{g - j}{j}\right] && \text{(51)}
\end{aligned}$$

Now, q may be simplified as follows, in which process both the numerator and denominator are divided by j/m, which does not affect the value of the fraction. The new numerator, $\frac{1 - v^{mtr}}{j/m}$, however, by the general formula $a_{\overline{n}|} = \frac{1 - v^n}{i}$ at i%, is equal to $a_{\overline{mtr}|}$ at j/m%; similarly with the new denominator:

$$q = \frac{v^{mf}}{r} \frac{1 - v^{mtr}}{1 - v^{mt}} = \frac{v^{mf}}{r} \frac{\frac{1 - v^{mtr}}{j/m}}{\frac{1 - v^{mt}}{j/m}} = \frac{v^{mf}}{r} \frac{a_{\overline{mtr}|}}{a_{\overline{mt}|}} \text{ at } j/m\%$$

Hence we have, substituting back in (51):

$$A = 1 + \left[1 - \frac{v^{mf}\, a_{\overline{mtr}|}}{r\, a_{\overline{mt}|}}\right] \left[\frac{g - j}{j}\right] \text{ at } j/m\%. \tag{52}$$

In particular, when the dividend is semi-annual m = 2, and (52) becomes:

$$A = 1 + \left[1 - \frac{v^{2f}\, a_{\overline{2tr}|}}{r\, a_{\overline{2t}|}}\right] \left[\frac{g - j}{j}\right] \text{ at } j/2\% \tag{53}$$

The formulas for the <u>premium</u> <u>k</u> of the loan are:

$$\text{When } m = m. \quad k = \left[1 - \frac{v^{mf}}{r} \frac{a_{\overline{mtr}|}}{a_{\overline{mt}|}}\right] \left[\frac{g - j}{j}\right] \tag{54}$$

$$\text{When } m = 2. \quad k = \left[1 - \frac{v^{2f}}{r} \frac{a_{\overline{2tr}|}}{a_{\overline{2t}|}}\right] \left[\frac{g - j}{j}\right] \tag{55}$$

where v^{mf}, $a_{\overline{mtr}|}$, $a_{\overline{mt}|}$, are at j/m%

and v^{2f}, $a_{\overline{2tr}|}$, $a_{\overline{2t}|}$, are at j/2%

Here the first payment matures in f years, the interval between equal payments is t years, the number of equal payments is r, and the total period of the loan is f + (r - 1)t = f - t + tr years. It will be noticed that formulas (54) and (55) involve the use of the v^n and $a_{\overline{n}|}$ tables, thus requiring us to

look up two different places in a set of interest tables. In extensive tables it would be more convenient to express the solution in terms of a single tabular function. To this end we proceed to express the solution in terms of the $a_{\overline{n}|}$ function. Returning to the value of q on p. 60, we have:

$$q = \frac{v^{mf}}{r}\,\frac{1 - v^{mtr}}{1 - v^{mt}} = \frac{v^{mf} - v^{mf}\,v^{mtr}}{r(1 - v^{mt})}$$

$$= \frac{1}{r}\,\frac{v^{mf} - v^{mf+mtr}}{1 - v^{mt}} = \frac{1}{r}\,\frac{v^{mf} - v^{m(f+tr)}}{1 - v^{mt}}.$$

By adding and subtracting 1 in the numerator,

$$= \frac{1}{r}\,\frac{1 - v^{m(f+tr)} - 1 + v^{mf}}{1 - v^{mt}}$$ Enclosing in brackets,

$$= \frac{1}{r}\,\frac{(1 - v^{m(f+tr)}) - (1 - v^{mf})}{(1 - v^{mt})}$$

Dividing both numerator and denominator by j/m:

$$= \frac{1}{r}\,\frac{\left[\frac{1 - v^{m(f+tr)}}{j/m} - \frac{1 - v^{mf}}{j/m}\right]}{\left[\frac{1 - v^{mt}}{j/m}\right]}$$

$$= \frac{1}{r}\,\frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{a_{\overline{mt}|}}$$ all at $j/m\%$.

Substituting this value of q back in formula (51), we have:

$$A = 1 + \left[1 - \frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{r a_{\overline{mt}|}}\right]\left[\frac{g - j}{j}\right] \text{ at } j/m\% \qquad (56)$$

This gives for the premium $k = A - 1$, the formula given at the outset of the discussion, namely (49); and, in particular, when $m = 2$, we have the formula (50). The formulas (49) and (50), it will be noticed, involve <u>only</u> the tabular function $a_{\overline{n}|}$, and all

at the rates j/m and $j/2$ respectively.

Having developed the formulas, we now turn to an application of the same to a few problems. Turning to the Baldwin Township Bonds, page 56, we have: $f = 5$, $t = 5$, $r = 6$, $g = .04$, $m = 2$, $C = 60000$. Hence, $m(f + tr) = 2(5 + 5 \times 6) = 70$; $mf = 2 \times 5 = 10$; $mt = 2 \times 5 = 10$. Substituting in (50) we have:

$$k = \left[1 - \frac{a_{\overline{70|}} - a_{\overline{10|}}}{6 \times a_{\overline{10|}}}\right] \left[\frac{.04 - j}{j}\right] \text{ at } j/2\%.$$

Let us first value the security at 3%; here $j = .03$, and the second factor in the above formula becomes:

$$\frac{.04 - .03}{.03} = \frac{.01}{.03} = \frac{1}{3}.$$

By Spitzer's Table, when $j/2 = 1\frac{1}{2}\%$:

$a_{\overline{70|}}$	=	43.15487183	(a)
$a_{\overline{10|}}$	=	9.22218455	(b)
$a_{\overline{70|}} - a_{\overline{10|}}$	=	33.93268728	(c)
Divide this by 6	=	5.65544788	(d)
Divide (d) by (b)	=	.61324384	(e)
Complement of (e) = 1 - (e)	=	.38675616	(f) = first facto
(f) x 1/3 = k	=	.12891872	(g)
$[1 + (g)] \times C = A$	=	67,735.123	
Hence, A	=	$67,735.12	

Here the <u>method</u> of computation is indicated in full detail. While Spitzer's Table gives the value of the $a_{\overline{n|}}$ functions to eight places, it will be found in practice that ordinarily from four to six places of decimals is sufficient. The 7-and 8-place tables

$j_{(2)}$ = means j is nominal rate. [illegible] semi-annual

are only needed when the value of the security is demanded to the nearest cent or mill.

*Example.--Find the value of this security in order to realize the following nominal rates of interest: $j_{(2)} = 3\frac{1}{4}, 3\frac{1}{2}, 3\frac{3}{4}, 4, 4\frac{1}{4}\%$. The following functions are given from Spitzer's Table to aid in the solution, viz:

Functions from Spitzer's Tables			Summary of Final Results		
%	$a_{\overline{70}\vert}$	$a_{\overline{10}\vert}$	j%	k	A
$1\frac{1}{2}$	43.15487183	9.22218455	.0300	.12891872	$67,735.123
$1\frac{5}{8}$	41.62679858	9.16140980	.0325	.09447265	65,668.359
$1\frac{3}{4}$	40.17790267	9.10122291	.0350	.06155807	63,963.484
$1\frac{7}{8}$	38.80334700	9.04161693	.0375	.03009291	61,805.575
2	37.49861929	8.98258501	.0400	.00000000	60,000.000
$2\frac{1}{8}$	36.25950881	8.92412033	.0425	-.02879323	58,272.406

*Example.--Bismark Bond offering, Commercial & Financial Chronicle, Vol. 73, p. 250. Viz.: $60,000 Bonds, dated Sept. 1, 1901, $4\frac{1}{2}\%$, semi-annual, maturing as follows: $20,000 each on Sept. 1, 1911; Sept. 1, 1916; Sept. 1, 1921. Value for $3\frac{1}{2}\%$, 4%, $4\frac{1}{2}\%$.

*Example.--Fort Lee Special School Bonds, Chronicle Vol. 76, p. 1262. Viz.: $18,000, $4\frac{1}{2}\%$, dated July 1, 1903, interest semi-annually, matures $1000 yearly on July 1st from 1908 to 1925 inclusive. Value for $j_{(2)} = 5\%, 4\frac{1}{2}\%, 4\%$.

Example.--Celina, Ohio. Bond Offering. Proposals will be received until 12 M., July 30, by Chas. R. Rohrer, Village Clerk,

for \$25000, 4% highway improvement bonds. Denomination \$500, dated June 20th, 1901. Interest, semi-annual. Principal will mature \$2500 yearly on June 20, from 1923 to 1932 inclusive. Accrued interest is to be paid by the purchaser.

Comm. and Fin. Chronicle, Vol. 73, p. 200.

Evaluate this security at the following nominal rates of interest: $j_{(2)} = 3\frac{1}{4}\%,\ 3\frac{1}{2}\%,\ 3\frac{3}{4}\%,\ 4\%,\ 4\frac{1}{4}\%$.

Example.--Celina, Ohio. Bond Sale. On July 30, the \$25000, 4% highway improvement bonds were awarded to W. R. Todd and Co., of Cincinnati, at 104.20 and accrued interest, an interest basis of about 3.75%. Chronicle, Vol. 73, p. 304. This example suggests the following general problem: having given the bid, or premium, on a security, to find the rate of interest the bidder is realizing on the purchase if his offer is accepted. The following suggestions are made to the student for obtaining an approximate solution. Compute the value of the security at two rates of interest in the neighborhood of the rate of interest realized by the purchaser and then by simple proportion the required approximate value can be determined.

Example.--The Bismarck Bond offering above made was taken by N. W. Harris and Co., of Chicago, at 1.0418 on the dollar; show that this bid corresponds to an interest basis of about 4.123%.

Example.--Baltimore, Md. Bond Offering. Proposals will be received until 12 M., Dec. 23, by David Ambach, President of the Commissioners of Finance, for \$1,000,000, $3\frac{1}{2}\%$ "Western Maryland Railroad 1952 Refunding Loan." Interest on these bonds will be

payable January 1 and July 1. Bonds are dated January 1, 1902. The city does not tax its own bonds and will pay the tax imposed on this issue by the State of Maryland. Chronicle, Vol. 73, p. 1226. Evaluate at 3% and $3\frac{1}{4}$%.

Example.--Baltimore, Md. Bond Sale. On Dec. 23, the \$1,000,000, $3\frac{1}{2}$%, "Western Railroad Refunding Loan" was awarded jointly to Hambleton & Co., Baltimore, and Dick Bros. & Co., and Kountze Bros. of New York City at 112.425. Show that this corresponds to an interest basis of about 3.017%. [illegible] 3.0178

Example.--Baldwin Township School District Bonds. \$60,000, 4% coupon, interest semi-annual. Date May 1, 1903, maturity \$10,000 on May 1, 1908, 1913, 1918, 1923, 1928, 1933. Chronicle, Vol. 76, p. 716. Evaluate at $j_{(2)}$ = 3%, $3\frac{1}{4}$%, $3\frac{1}{2}$%, $3\frac{3}{4}$%, 4%, $4\frac{1}{4}$%.

Example.--Saginaw, Michigan. Bond Offering. \$10,000. Interest semi-annual, $3\frac{1}{2}$%. Date 1903, matures 1923. Chronicle, March, 1903. Evaluate at 3%, 4%, 5%.

Example.--Mansfield, Ohio. Bond Offering. \$40,000. Interest semi-annual, 4%, payable March 15 and September 15. Matures \$4000 annually, 1905 to 1914 inclusive. Date September 15, 1903. Chronicle, March, 1903.

Evaluate at 3% and 5%.

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L E S S O N XI.

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Life Annuities and Insurance.

In developing the theory of life contingencies it is necessary first to call the attention of the student to the Mortality Table at the base of all the results eventually obtained. A Mortality Table may be said to be given when out of a group of persons born the number of survivors after one, two, three, etc., years are given. For example, the H^m (healthy males) British Table runs as follows:

x = age	l_x = number living at age x.
0	127283
1	112925
2	108963
-	- -
-	- -
10	100000
-	- -
-	- -
50	72795
-	- -
-	- -
75	25602
-	- -
-	- -
90	1273
-	- -
-	- -
100	4
101	1
102	0

For parts omitted, see "Short Collection of Acturial Tables," or the "Institute of Actuaries' Text Book, Part II."

The Mortality Tables in use in this country by legal reserve life insurance companies are: (a) the American Experience, tabulated at the end of this lesson, and (b) The Combined Experience or

Actuaries'. Both of these Tables, with very extensive derived or auxiliary Tables, are given in "The Principles and Practice of Life Insurance," by Nathan Willey, published by the Spectator Company, New York City. It is not our purpose in this course to show how Mortality Tables are derived, but rather to show how the actuarial theory is developed after the fundamental Mortality Table is assumed. Suffice to say that the derivation of Tables of human mortality from vital statistics, such as the experience of insurance companies, census returns, etc., constitutes in itself an extensive and difficult mathematical theory.

There are various useful "derived columns" (i. e., columns which can be obtained by computation from the l_x column) which usually appear in the Mortality, or Life, Table. For example:

x	l_x	d_x	p_x	q_x	$\mathring{e}_x$	x
25	93,044	658	.99,293	.00707	38.382	25
26	92,386	664	.99,280	.00720	37.652	26
27	91,722	673	.99,268	.00732	36.921	27
28	91,049	678	.99,254	.00746	36.189	28
29	90,371	686	.99,241	.00759	35.458	29

We have here a section of the H^m (graduated) Table, giving five functions, l_x, d_x, p_x, q_x, and $\mathring{e}_x$. All those after l_x can be derived from l_x. The symbol d_x represents the number among the survivors l_x who die before attaining the age $x + 1$. Hence,

$$d_x = l_x - l_{x+1}. \tag{57}$$

For example,

$$d_{25} = l_{25} - l_{26} = 93044 - 92386 = 658.$$

That is, out of 93044 persons who reach the age of 25 the Table shows that 658 fail to reach the age of 26. The symbol p_x is the ratio of those attaining age x + 1 to those attaining age x, hence,

$$p_x = \frac{l_{x+1}}{l_x} \tag{58}$$

For example,

$$p_{25} = \frac{l_{26}}{l_{25}} = \frac{92386}{93044} = .99293.$$

It is, therefore, the "rate of survival," or the "probability of living." The latter term is derived from the consideration that out of l_x persons starting out at age x with equal chances, l_{x+1} persons actually reach the age x + 1, or survive one year. Hence any individual may consider that he has l_{x+1} chances out of l_x for living one year. Now, the ratio of the favorable events or cases (living) to the total events (living and dying) is defined as the "probability of living":

$$\left.\begin{array}{l}\text{Probability of}\\ \text{living one year}\end{array}\right\} = \frac{\text{Number surviving one year}}{\text{Number surviving one year plus number dying in year}} = \frac{l_{x+1}}{l_x}$$

Transposing (57) we have,

$$l_x = l_{x+1} + d_x$$

Hence, $$p_x = \frac{l_{x+1}}{l_x} = \frac{l_{x+1}}{l_{x+1} + d_x}$$

which agrees with the preceding verbal formula. The symbol q_x represents the ratio of those dying in the age x to those who start out at age x.

Hence, $$q_x = \frac{d_x}{l_x} = \frac{l_x - l_{x+1}}{l_x} = \frac{d_x}{l_{x+1} + d_x} \tag{59}$$

For example,

$$q_{25} = \frac{d_{25}}{l_{25}} = \frac{658}{93044} = .00707.$$

It is evident that from the definition q_x is the "average death rate" for the age x. It is also said to be the "probability of dying" during the year of age x. Out of l_x persons of age x, starting out at age x with equal chances, d_x persons fail to reach the age x + 1, or die within one year. Hence, any individual may consider that he has d_x chances out of l_x for dying within one year. Now, the ratio of the number of cases in which the event happens (death in d_x cases) to the total number of cases (living and dying = $l_{x+1} + d_x = l_x$) is defined as the "probability of dying." Thus:

$$\left.\begin{array}{r}\text{Probability of dying}\\ \text{in one year}\end{array}\right\} = \frac{\text{Number dying in year}}{\begin{array}{l}\text{Number surviving one year}\\ \text{plus number dying in year}\end{array}} = \frac{d_x}{l_x}$$

It will be seen that $p_x + q_x = 1,$ for,

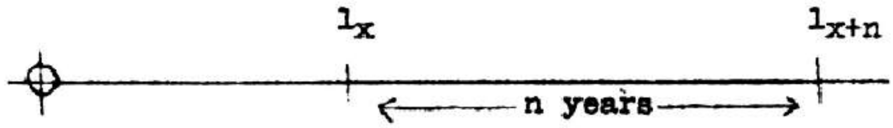
$$p_x + q_x = \frac{l_{x+1}}{l_x} + \frac{d_x}{l_x} = \frac{l_{x+1} + d_x}{l_x} = \frac{l_x}{l_x} = 1.$$

$$\therefore\ p_x = 1 - q_x \quad \text{and} \quad q_x = 1 - p_x.$$

(It would be advantageous for the student at this point to read a few pages in any college algebra on the "Elements of the Theory of Probability." Wentworth's College Algebra (revised), Chap. XXIII, p. 276, Chrystal's Algebra, Part II., Chap. XXXVI, p. 538, are suggested, and will be found in almost any good reference library.)

Pure endowment means if [illegible] lives [illegible] no. yrs, get money - if they die they lose it. [illegible] endowment & insurance.

Our first problem will be to find the value of a pure endowment of 1 issued to a person aged x, and due in n years. The symbol for this benefit is ${}_nE_x$. Here E refers to the pure endowment, x to the age of the person on whose life the contingent benefit depends, n the number of years elapsing before the endowment matures.



Suppose a company should undertake to sell a pure endowment of one unit to each of l_x persons, the endowment to mature in n years: what ought each individual to pay the company for the endowment ${}_nE_x$? By the Mortality Table there will be living after n years l_{x+n} persons of the original l_x who took endowments. Hence, the payment to each individual then living being 1, the total payment which must be provided for is l_{x+n} units. Therefore the company must have in hand at the present moment a sum which at the assumed rate of interest i will accumulate to l_{x+n}. The present value of l_{x+n} due in n years is $v^n l_{x+n}$. Since each of the l_x individuals must contribute equally to this fund, we have:

$$ {}_nE_x = \frac{v^n l_{x+n}}{l_x} \qquad (60)$$

If we multiply both numerator and denominator by v^x, we have,

$$ {}_nE_x = \frac{v^x v^n l_{x+n}}{v^x l_x} = \frac{v^{x+n} l_{x+n}}{v^x l_x}$$

If we then let

$$v^x \; l_x = D_x$$

$$v^{x+1} \; l_{x+1} = D_{x+1}$$

$$\text{--} \quad \text{--} \qquad \text{--}$$

$$v^{x+n} \; l_{x+n} = D_{x+n} \qquad \text{it follows that}$$

$$_nE_x = \frac{D_{x+n}}{D_x} \tag{61}$$

The symbol $D_x = v^x l_x$ is called a "commutation symbol," and may be computed once for all for any given Mortality Table and rate of interest. It is given for the American Experience Mortality Table at 3% on p. 76.

Problem.--What is the value of a pure endowment of \$5000 due in 20 years, issued to a person aged 35 years?

Here $n = 20$, $x = 35$, $x + n = 55$, assumed $i = 3\%$. Using American Experience table, p. 76, we have,

$$_{20}E_{35} = \frac{D_{55}}{D_{35}} = \frac{1270.388}{2907.819} = .43689.$$

This is the value of a similar endowment of 1; hence the value of an endowment of 5000 would be 5000 x .43689 = 2184.45. The value of the benefit, therefore, is \$2184.45. The problem may also be worked by formula (60), using only the l_x and v^n Tables; this is left as an exercise for the student.

We next turn our attention to the evaluation of the ordinary immediate life annuity. The symbol used for this is a_x. It denotes the present value of an immediate annuity, with annual rent of 1, issued to a person aged x, and to continue throughout life. The status here is "the life of the annuitant."

l_x $\quad l_{x+1}$ $\quad l_{x+2}$ $\quad l_{x+3}$ --- etc., to end of table

x yrs. $\quad 1 \quad 1 \quad 1$

An annuity, as above described, is issued to each of l_x persons, to find the value, a_x, of each annuity. At the end of the first year there will be l_{x+1} survivors, each of whom is to receive 1, a total of l_{x+1}. After two years l_{x+2} units must be paid out to the survivors; and so on to the end of the table, when there are no lives surviving. In order to meet these maturing contracts there must be on hand at the present moment:

the present value of l_{x+1}, due in 1 year, $= v\, l_{x+1}$,

the present value of l_{x+2}, due in 2 years, $= v^2 l_{x+2}$,

the present value of l_{x+3}, due in 3 years, $= v^3 l_{x+3}$,

etc., to end of table,

where v is taken at the assumed rate of interest i. Adding these sums together we have,

$$(v l_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \ldots \text{to end of table}).$$

Each one of the l_x individuals must contribute the same amount toward this fund, whence,

$$a_x = \frac{v l_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \ldots \text{ to end of table.}}{l_x}$$

Multiplying numerator and denominator by v^x we have,

$$a_x = \frac{v^{x+1} l_{x+1} + v^{x+2} l_{x+2} + v^{x+3} l_{x+3} + \ldots \text{ to end of table,}}{v^x\, l_x}$$

$$= \frac{D_{x+1} + D_{x+2} + D_{x+3} + \ldots \text{ to end of table.}}{D_x}$$

We now introduce another commutation symbol, N_x.

Let $N_x = D_{x+1} + D_{x+2} + D_{x+3} + \ldots$ to end of table.

Whence the simple formula,

$$a_x = \frac{N_x}{D_x}. \tag{62}$$

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Age $= x$	Number living $= l_x$	Number dying $= d_x$	Death rate $= q_x = \frac{d_x}{l_x}$	Expectation of life $= \mathring{e}_x$	Age $= x$
10	100,000	749	.007490	48.72	10
11	99,251	746	.007516	48.09	11
12	98,505	743	.007543	47.45	12
13	97,762	740	.007569	46.80	13
14	97,022	737	.007596	46.16	14
15	96,285	735	.007634	45.51	15
16	95,550	732	.007661	44.85	16
17	94,818	729	.007688	44.19	17
18	94,089	727	.007727	43.53	18
19	93,362	725	.007765	42.87	19
20	92,637	723	.007805	42.20	20
21	91,914	722	.007855	41.53	21
22	91,192	721	.007906	40.85	22
23	90,471	720	.007958	40.17	23
24	89,751	719	.008011	39.49	24
25	89,032	718	.008065	38.81	25
26	88,314	718	.008130	38.12	26
27	87,596	718	.008197	37.43	27
28	86,878	718	.008264	36.73	28
29	86,160	719	.008345	36.03	29
30	85,441	720	.008427	35.33	30
31	84,721	721	.008510	34.63	31
32	84,000	723	.008607	33.92	32
33	83,277	726	.008718	33.21	33
34	82,551	729	.008831	32.50	34
35	81,822	732	.008946	31.78	35
36	81,090	737	.009089	31.07	36
37	80,353	742	.0092[illegible]4	30.35	37
38	79,611	749	.009408	29.63	38
39	78,862	756	.009586	28.90	39
40	78,106	765	.009794	28.18	40
41	77,341	774	.010008	27.45	41
42	76,567	785	.010252	26.72	42
43	75,782	797	.010517	25.99	43
44	74,985	812	.010829	25.27	44
45	74,173	828	.011163	24.54	45
46	73,345	848	.011562	23.81	46
47	72,497	870	.012000	23.08	47
48	71,627	896	.012509	22.35	48
49	70,731	927	.013106	21.63	49
50	69,804	962	.013781	20.91	50
51	68,842	1,001	.014541	20.20	51
52	67,841	1,044	.015389	19.49	52
53	66,797	1,091	.016333	18.79	53
54	65,706	1,143	.017396	18.09	54

x	l_x	d_x	$q_x = \frac{d_x}{l_x}$	$\mathring{e}_x$	x
55	64,563	1,199	.018571	17.40	55
56	63,364	1,260	.019885	16.72	56
57	62,104	1,325	.021335 16.07	16.05	57
58	60,779	1,394	.022936 15.34	15.39	58
59	59,385	1,468	.024720 14.[illegible]	14.74	59
60	57,917	1,546	.026693	14.10	60
61	56,371	1,628	.028880	13.47	61
62	54,743	1,713	.031292	12.86	62
63	53,030	1,800	.033943	12.26	63
64	51,230	1,889	.036873	11.67	64
65	49,341	1,980	.040129	11.10	65
66	47,361	2,070	.043707	10.54	66
67	45,291	2,158	.047647	10.00	67
68	43,133	2,243	.052002	9.47	68
69	40,890	2,321	.056762	8.97	69
70	38,569	2,391	.061993	8.48	70
71	36,178	2,448	.067665	8.00	71
72	33,730	2,487	.073733	7.55	72
73	31,243	2,505	.080178	7.11	73
74	28,738	2,501	.087028	6.68	74
75	26,237	2,476	.094371	6.27	75
76	23,761	2,431	.102311	5.88	76
77	21,330	2,369	.111064	5.49	77
78	18,961	2,291	.120827	5.11	78
79	16,670	2,196	.131734	4.75	79
80	14,474	2,091	.144466	4.39	80
81	12,383	1,964	.158605	4.05	81
82	10,419	1,816	.174297	3.71	82
83	8,603	1,648	.191561	3.39	83
84	6,955	1,470	.211359	3.08	84
85	5,485	1,292	.235550	2.77	85
86	4,193	1,114	.265681	2.47	86
87	3,079	933	.303020	2.18	87
88	2,146	744	.346692	1.91	88
89	1,402	555	.395863	1.66	89
90	847	385	.454545	1.42	90
91	462	246	.532468	1.19	91
92	216	137	.634259	.98	92
93	79	58	.734177	.80	93
94	21	18	.857143	.64	94
95	3	3	1.000000	.50	95

Age	v*t+ D_x	N_x	M_x	Age
10	7440.940	181134.595	2165.17505	10
11	7170.105	173693.65[illegible]	2111.06569	11
12	6908.945	166523.650	2058.74274	12
13	6657.116	159614.605	2008.14806	13
14	6414.297	152957.489	1959.22534	14
15	6180.169	146543.192	1911.92010	15
16	5954.361	140363.023	1866.11733	16
17	5736.645	134408.662	1821.83013	17
18	5526.738	128672.017	1779.00906	18
19	5324.305	123145.279	1737.54926	19
20	5129.087	117820.974	1697.40776	20
21	4940.831	112691.887	1658.54295	21
22	4759.242	107751.056	1620.86231	22
23	4584.091	102991.814	1584.32983	23
24	4415.156	98407.723	1548.91060	24
25	4252.218	93992.567	1514.57076	25
26	4095.075	89740.349	1481.27748	26
27	3943.477	85645.274	1448.95391	27
28	3797.236	81701.797	1417.57180	28
29	3656.169	77904.561	1387.10373	29
30	3520.056	74248.392	1357.48188	30
31	3388.732	70728.336	1328.68280	31
32	3262.032	67339.604	1300.68369	32
33	3139.761	64077.572	1273.42469	33
34	3021.738	60937.811	1246.84983	34
35	2907.819	57916.073	1220.94238	35
36	2797.868	55008.254	1195.68600	36
37	2691.688	52210.386	1170.99775	37
38	2589.158	49518.698	1146.86597	38
39	2490.096	46929.540	1123.21603	39
40	2394.393	44439.444	1100.04033	40
41	2301.885	42045.051	1077.27179	41
42	2212.475	39743.166	1054.90634	42
43	2126.011	37530.691	1032.88371	43
44	2042.380	35404.680	1011.17568	44
45	1961.420	33362.300	989.70326	45
46	1883.034	31400.880	968.44548	46
47	1807.051	29517.846	947.30823	47
48	1733.365	27710.795	926.25443	48
49	1661.827	25977.430	905.20288	49

Age	D_x	N_x	M_x	Age
50	1592.279	24315.603	884.05736	50
51	1524.597	22723.324	862.75260	51
52	1458.668	21198.727	841.22981	52
53	1394.389	19740.059	819.43628	53
54	1331.665	18345.670	797.32496	54
55	1270.388	17014.005	774.83447	55
56	1210.481	15743.617	751.92924	56
57	1151.855	14533.136	728.55978	57
58	1094.447	13381.281	704.70052	58
59	1038.199	12286.834	680.32990	59
60	983.0434	11248.6346	655.41308	60
61	928.9342	10265.5912	629.93663	61
62	875.8318	9336.6570	603.89030	62
63	823.7142	8460.8252	577.28229	63
64	772.5777	7637.1110	550.13727	64
65	722.4177	6864.5333	522.47981	65
66	673.2309	6142.1156	494.33435	66
67	625.0544	5468.8847	465.76658	67
68	577.9344	4843.8303	436.85178	68
69	531.9228	4265.8959	407.67343	69
70	487.1163	3733.9731	378.35982	70
71	443.6104	3246.8568	349.04166	71
72	401.5468	2803.2464	319.89886	72
73	361.1065	2401.6996	291.15412	73
74	322.4793	2040.5931	263.04462	74
75	285.8396	1718.1138	235.79742	75
76	251.3250	1432.2742	209.60826	76
77	219.0406	1180.9492	184.64400	77
78	189.0417	961.9086	161.02499	78
79	161.3596	772.8669	138.84893	79
80	136.0225	611.5073	118.21156	80
81	112.9824	475.4848	99.133301	81
82	92.29402	362.50242	81.735711	82
83	73.98783	270.20840	66.117681	83
84	58.07247	196.22057	52.357301	84
85	44.46441	138.14810	40.440681	85
86	33.00074	93.68369	30.272081	86
87	23.52725	60.68295	21.759785	87
88	15.92040	37.15570	14.838195	88
89	10.09799	21.23530	9.479489	89
90	5.922884	11.137306	5.598496	90
91	3.136567	5.214422	2.984691	91
92	1.423735	2.077855	1.363215	92
93	.5055512	.6541202	.4864994	93
94	.1304729	.1485690	.1261457	94
95	.0180961	.0180961	.0175690	95

Age	D_x	N_x	M_x	Age
10	3.8716278	5.2580014	3.3354931	10
11	.8555255	.2397840	.3245018	11
12	.8394117	.2214755	.3136021	12
13	.8232862	.2030726	.3027956	13
14	.8071491	.1845708	.2920843	14
15	.7910003	.1659657	.2814697	15
16	.7748352	.1472527	.2709388	16
17	.7586580	.1284274	.2605078	17
18	.7424689	.1094841	.2501781	18
19	.7262629	.0904179	.2399370	19
20	.7100401	.0712227	.2297862	20
21	.6938000	.0518928	.2197268	21
22	.6775378	.0324217	.2097460	22
23	.6612533	.0128027	.1998456	23
24	.6449460	4.9930292	.1900265	24
25	.6286155	.9730935	.1802897	25
26	.6122618	.9529877	.1706363	26
27	.5958793	.9327034	.1610546	27
28	.5794676	.9122316	.1515451	28
29	.5630262	.8915629	.1421090	29
30	.5465496	.8706870	.1327340	30
31	.5300372	.8495934	.1234214	31
32	.5134382	.8282705	.1141717	32
33	.4968967	.8067061	.1049733	33
34	.4802567	.7848869	.0958143	34
35	.4635673	.7627991	.0866951	35
36	.4463273	.7404279	.0776171	36
37	.4300248	.7177569	.0685562	37
38	.4131586	.6947693	.0595127	38
39	.3962161	.6714464	.0504633	39
40	.3791955	.6477686	.0414085	40
41	.3620836	.6237149	.0323253	41
42	.3448782	.5992625	.0232138	42
43	.3275655	.5743866	.0140516	43
44	.3101366	.5490607	.0048267	44
45	.2925707	.5232560	2.9955050	45
46	.2748582	.4969417	.9860752	46
47	.2569705	.4700847	.9764913	47
48	.2388901	.4426489	.9667303	48
49	.2205858	.4145962	.9567459	49

Age	$v^x l_x$ D_x	$\mathbb{N}_x$	M_x	Age
50	3.2020191	4.3858850	2.9464804	50
51	.1831551	.3564718	.9358863	51
52	.1639566	.3263099	.9249147	52
53	.1443841	.2953484	.9135153	53
54	.1243949	.2635336	.9016354	54
55	.1039364	.2308064	.8892089	55
56	.0829581	.1971046	.8761769	56
57	.0613978	.1623595	.8624653	57
58	.0391946	.1264977	.8480046	58
59	.0162806	.0894400	.8327196	59
60	2.9925727	.0510997	.8165152	60
61	.9679850	.0113840	.7992968	61
62	.9424207	3.9701915	.7809580	62
63	.9157765	.9274127	.7613882	63
64	.8879421	.8829292	.7404710	64
65	.8587883	.8366110	.7180696	65
66	.8281641	.7883180	.6940208	66
67	.7959179	.7378987	.6681684	67
68	.7618785	.6851889	.6403341	68
69	.7258486	.6300103	.6103124	69
70	.6876327	.5721712	.5779050	70
71	.6470017	.5114631	.5428773	71
72	.6037362	.4476612	.5050128	72
73	.5576353	.3805188	.4641230	73
74	.5085019	.3097564	.4200294	74
75	.4561224	.2350519	.3725390	75
76	.4002357	.1560261	.3214085	76
77	.3405246	.0722312	.2663352	77
78	.2765577	2.9831338	.2068933	78
79	.2077949	.8881047	.1425424	79
80	.1336107	.7864017	.0726601	80
81	.0530107	.6771366	1.9962195	81
82	1.9651736	.5593109	.9124119	82
83	.8691603	.4316988	.8203176	83
84	.7639703	.2927446	.7189773	84
85	.6480125	.1403449	.6068184	85
86	.5185236	1.9716640	.4810422	86
87	.3715712	.7830668	.3376545	87
88	.2019540	.5700255	.1713809	88
89	.0042350	.3270585	0.9767849	89
90	0.7725332	.0467803	.7480714	90
91	.4964546	0.7172062	.4748994	91
92	.1534292	.3176153	.1345644	92
93	$\bar{1}$.7037652	$\bar{1}$.8156575	$\bar{1}$.6870823	93
94	.1155202	.1719282	.1008724	94
95	$\bar{2}$.2575850	$\bar{2}$.2575850	$\bar{2}$.2447477	95

Age	C_x	R_x	S_x	Age
10	54.10936	73421.60266	3698145.914	10
11	52.32295	71256.42761	3517011.319	11
12	50.59468	69145.36192	3343317.664	12
13	48.92272	67086.61918	3176794.114	13
14	47.30524	65078.47112	3017179.509	14
15	45.80277	63119.24578	2864222.020	15
16	44.28720	61207.32568	2717678.828	16
17	42.82107	59341.20835	2577315.805	17
18	41.45980	57519.37822	2442907.143	18
19	40.14150	55740.36916	2314235.126	19
20	38.86481	54002.81990	2191089.847	20
21	37.68064	52305.41214	2073268.873	21
22	36.53248	50646.86919	1960576.986	22
23	35.41923	49026.00688	1852825.930	23
24	34.33984	47441.67705	1749834.116	24
25	33.29328	45892.76645	1651426.393	25
26	32.32357	44378.19569	1557433.826	26
27	31.38211	42896.91821	1467693.477	27
28	30.46807	41447.96430	1382048.203	28
29	29.62185	40030.39250	1300346.406	29
30	28.79908	38643.28877	1222441.845	30
31	27.99911	37285.80689	1148193.453	31
32	27.25900	35957.12409	1077465.117	32
33	26.57486	34656.44040	1010125.513	33
34	25.90745	33383.01571	946047.941	34
35	25.25638	32136.16588	885110.130	35
36	24.68825	30915.22350	827194.057	36
37	24.13178	29719.53750	772185.803	37
38	23.64994	28548.53975	719975.417	38
39	23.17570	27401.67378	670456.719	39
40	22.76854	26278.45775	623527.179	40
41	22.36545	25178.41742	579087.735	41
42	22.02263	24101.14563	537042.684	42
43	21.70803	23046.23929	497299.518	43
44	21.47242	22013.35558	459768.827	44
45	21.25778	21002.17990	424364.147	45
46	21.13715	20012.47664	391001.847	46
47	21.05390	19044.03116	359600.967	47
48	21.05155	18096.72283	330083.121	48
49	21.14552	17170.46840	302372.326	49

Age	C_x	R_x	S_x	Age
50	21.30476	16265.26552	276394.896	50
51	21.52279	15381.20816	252079.293	51
52	21.79353	14518.45556	229355.969	52
53	22.11132	13677.22575	208157.242	53
54	22.49049	12857.78947	188417.183	54
55	22.90523	12060.46451	170071.513	55
56	23.36946	11285.63004	153057.508	56
57	23.85926	10533.70080	137313.891	57
58	24.37062	9805.14102	122780.755	58
59	24.91682	9100.44050	109399.474	59
60	25.47645	8420.11060	97112.6401	60
61	26.04633	7764.69752	85864.0055	61
62	26.60801	7134.76089	75598.4143	62
63	27.14502	6530.87059	66261.7573	63
64	27.65746	5953.58830	57800.9321	64
65	28.14546	5403.45103	50163.8211	65
66	28.56777	4880.97122	43299.2878	66
67	28.91480	4386.63687	37157.1722	67
68	29.17835	3920.87029	31688.2875	68
69	29.31361	3484.01851	26844.4572	69
70	29.31816	3076.34508	22578.5613	70
71	29.14280	2697.98526	18844.5882	71
72	28.74474	2348.94360	15597.7314	72
73	28.10950	2029.04474	12794.4850	73
74	27.24720	1737.89062	10392.7854	74
75	26.18916	1474.84600	8352.1923	75
76	24.96426	1239.04858	6634.0785	76
77	23.61901	1029.44032	5201.8043	77
78	22.17606	844.79632	4020.8551	78
79	20.63737	683.77133	3058.9465	79
80	19.07826	544.92240	2286.0796	80
81	17.39759	426.710841	1674.5723	81
82	15.61803	327.577540	1199.08750	82
83	13.76038	245.841829	836.58508	83
84	11.91662	179.724148	566.37668	84
85	10.16860	127.366847	370.15611	85
86	8.512296	86.926166	232.00801	86
87	6.921590	56.654085	138.32432	87
88	5.358706	34.894300	77.64137	88
89	3.880993	20.056105	40.48567	89
90	2.613805	10.576616	19.250368	90
91	1.621476	4.978120	8.113062	91
92	.8767154	1.993429	2.898640	92
93	.3603537	.6302141	.8207853	93
94	.1085767	.1437147	.1666651	94
95	.0175690	.0175690	.0180961	95

Age	C_x	R_x	S_x	Age
10	1.7332724	4.8658238	6.5679241	10
11	.7186922	.8528240	.5461737	11
12	.7041049	.8397631	.5241776	12
13	.6895106	.8266359	.5019690	13
14	.6749092	.8134374	.4796012	14
15	.6608918	.8001618	.4570066	15
16	.6462783	.7868034	.4341981	16
17	.6316575	.7733565	.4111677	17
18	.6176272	.7598142	.3879069	18
19	.6035936	.7461698	.36440740	19
20	.5895566	.7324164	.3406602	20
21	.5761183	.7185466	.3166556	21
22	.5626792	.7045527	.2923838	22
23	.5492392	.6904266	.2678346	23
24	.5357983	.6761500	.2429963	24
25	.5223566	.6617444	.2178592	25
26	.5095194	.6471697	.1924097	26
27	.4966822	.6324261	.1666352	27
28	.4838449	.6175032	.1405231	28
29	.4716122	.6023898	.1140590	29
30	.4593786	.5870740	.0872282	30
31	.4471442	.5715435	.0600149	31
32	.4355099	.5557849	.0324032	32
33	.4244710	.5397839	.0043756	33
34	.4134247	.5235257	5.9759131	34
35	.4023711	.5069941	.9469972	35
36	.3924902	.4901724	.9176074	36
37	.3825894	.4730421	.8877217	37
38	.3738301	.4555839	.8573176	38
39	.3650329	.4377770	.8263708	39
40	.3573352	.4195999	.7948554	40
41	.3495776	.4010284	.7627444	41
42	.3428691	.3820378	.7300083	42
43	.3366205	.3626001	.6966179	43
44	.3318809	.3426863	.6625394	44
45	.3275180	.3222645	.6277386	45
46	.3250464	.3013009	.5921788	46
47	.3233326	.2797589	.5558209	47
48	.3232840	.2575999	.5186238	48
49	.3252185	.2347822	.4805420	49

Age	C_x	R_x	$\mathcal{C}_x$	Age
50	1.3284767	4.2112613	5.4415300	50
51	.3328985	.1869905	.4015372	51
52	.3383276	.1619206	.3605100	52
53	.3446147	.1359981	.3183914	53
54	.3519989	.1091663	.2751205	54
55	.3599347	.0813639	.2306316	55
56	.3686487	.0525259	.1848546	56
57	.3776569	.0225810	.1377145	57
58	.3868666	3.9914538	.0891304	58
59	.3964927	.9590624	.0390153	59
60	.4061388	.9253177	4.9872758	60
61	.4157465	.8901245	.9338111	61
62	.4250123	.8533795	.8785127	62
63	.4336902	.8149712	.8212630	63
64	.4418124	.7747788	.7619348	64
65	.4494084	.7326712	.7003907	65
66	.4558763	.6885062	.6364808	66
67	.4611202	.6421317	.5700426	67
68	.4650608	.5933825	.5008987	68
69	.4670694	.5420805	.4288547	69
70	.4671367	.4880350	.3536962	70
71	.4645313	.4310395	.2751867	71
72	.4585584	.3708726	.1930614	72
73	.4488531	.3072917	.1070226	73
74	.4353219	.2400225	.0167322	74
75	.4181216	.1687467	3.9218005	75
76	.3973187	.0930885	.8217806	76
77	.3732616	.0126011	.7161540	77
78	.3458844	2.9267520	.6043184	78
79	.3146544	.8349108	.4855718	79
80	.2805388	.7363346	.3590914	80
81	.2404891	.6301336	.2239038	81
82	.1936262	.5153141	.0788507	82
83	.1386304	.3906557	2.9225101	83
84	.0761532	.2546063	.7531053	84
85	.0072612	.1050564	.5683849	85
86	0.9300467	1.9391505	.3655030	86
87	.8402059	.7532312	.1408985	87
88	.7290599	.5427545	1.8900932	88
89	.5389428	.3022465	.6073012	89
90	.4172733	.0243469	.2844391	90
91	.2099105	0.6970653	0.9091848	91
92	$\bar{1}$.9428587	.2996009	.4621943	92
93	.5567289	$\bar{1}$.7994882	$\bar{1}$.9142297	93
94	.0357363	.1575012	.2218448	94
95	$\bar{2}$.2447478	$\bar{2}$.2447478	$\bar{2}$.2575850	95

Age = x	Net present value of life annuity-due of 1 per annum		Net premium for whole life insurance of 1000		Age = x
	$\text{a}_x = 1 + a_x$	Logarithm of $\text{a}_x = 1 + a_x$	Single premium = $1000A_x$	Annual premium = $1000P_x$	
10	24.34297	1.3863736	290.9815	11.953	10
11	24.22471	.3842585	294.4263	12.154	11
12	24.10259	.3820638	297.9823	12.363	12
13	23.97653	.3797864	301.6542	12.581	13
14	23.84634	.3774217	305.4466	12.809	14
15	23.71185	.3749654	309.3637	13.047	15
16	23.57314	.3724175	313.4023	13.295	16
17	23.42984	.3697694	317.5775	13.554	17
18	23.28173	.3670152	321.8912	13.826	18
19	23.12890	.3641550	326.3428	14.110	19
20	22.97114	.3611826	330.9376	14.407	20
21	22.80829	.3580928	335.6810	14.717	21
22	22.64039	.3548839	340.5714	15.043	22
23	22.46722	.3515494	345.6148	15.383	23
24	22.28862	.3480831	350.8169	15.740	24
25	22.10437	.3444781	356.1838	16.114	25
26	21.91421	.3407259	361.7217	16.506	26
27	21.71822	.3368241	367.4306	16.918	27
28	21.51612	.3327640	373.3168	17.351	28
29	21.30770	.3285367	379.3874	17.805	29
30	21.09296	.3241376	385.6420	18.283	30
31	20.87162	.3195562	392.0885	18.786	31
32	20.64345	.3147823	398.7342	19.315	32
33	20.40842	.3098094	405.5801	19.873	33
34	20.16648	.3046301	412.6270	20.461	34
35	19.91736	.2992318	419.8825	21.081	35
36	19.66077	.2936006	427.3560	21.736	36
37	19.39689	.2877321	435.0422	22.428	37
38	19.12541	.2816107	442.9494	23.160	38
39	18.84648	.2752302	451.0734	23.934	39
40	18.55979	.2685731	459.4232	24.754	40
41	18.26549	.2616313	467.9954	25.622	41
42	17.96323	.2543843	476.7993	26.543	42
43	17.65310	.2468211	485.8320	27.521	43
44	17.33500	.2389241	495.0967	28.561	44
45	17.00925	.2306853	504.5850	29.665	45
46	16.67568	.2220835	514.3006	30.841	46
47	16.33481	.2131142	524.2287	32.093	47
48	15.98670	.2037588	534.3677	33.426	48
49	15.63185	.1940104	544.7034	34.846	49

x	$\ddot{a}_x = 1 + a_x$	Logarithm of $\ddot{a}_x = 1 + a_x$	1000 A_x	1000 P_x	x
50	15.27095	1.1838659	555.2151	36.358	50
51	14.90448	.1733187	565.8890	37.968	51
52	14.53293	.1623533	576.7108	39.683	52
53	14.15677	.1509643	587.6669	41.511	53
54	13.77649	.1391387	598.7430	43.461	54
55	13.39276	.1268700	609.9194	45.541	55
56	13.00608	.1141465	621.1819	47.761	56
57	12.61716	.1009617	632.5101	50.131	57
58	12.22653	.0873031	643.8876	52.663	58
59	11.83476	.0731594	655.2983	55.371	59
60	11.44266	.0585270	666.7185	58.266	60
61	11.05093	.0433990	678.1282	61.364	61
62	10.66033	.0277707	689.5048	64.680	62
63	10.27155	0116362	700.8283	68.230	63
64	9.885235	0.9949870	712.0802	72.035	64
65	9.502167	.9778227	723.2382	76.113	65
66	9.123340	.9601539	734.2715	80.483	66
67	8.749452	.9419809	745.1617	85.167	67
68	8.381280	.9233104	755.8847	90.187	68
69	8.019767	.9041617	766.4145	95.566	69
70	7.665465	.8845385	776.7340	101.329	70
71	7.319163	.8644614	786.8204	107.501	71
72	6.981118	.8439250	796.6665	114.117	72
73	6.650948	.8228835	806.2834	121.228	73
74	6.327826	.8012545	815.6946	128.906	74
75	6.010761	.7789295	824.9290	137.242	75
76	5.698893	.7557904	834.0129	146.346	76
77	5.391461	.7317066	842.9670	156.352	77
78	5.088340	.7065761	851.7960	167.402	78
79	4.789717	.6803098	860.4934	179.654	79
80	4.495634	.6527910	869.0592	193.312	80
81	4.208487	.6241259	877.4226	208.489	81
82	3.927690	.5941373	885.6014	225.476	82
83	3.652064	.5625384	893.6290	244.691	83
84	3.378892	.5287743	901.5856	266.829	84
85	3.106936	.4923324	909.5066	292.734	85
86	2.838837	.4531404	917.3152	323.131	86
87	2.579263	.4114956	924.8754	358.581	87
88	2.333843	.3680715	932.0236	399.351	88
89	2.102925	.3228235	938.7500	446.402	89
90	1.880386	.2742471	945.2314	502.679	90
91	1.662462	.2207516	951.5790	572.391	91
92	1.459440	.1641861	957.4922	656.068	92
93	1.293875	.1118923	962.3147	743.746	93
94	1.138697	.0564080	966.8349	849.071	94
95	1.000000	.0000000	970.8734	970.873	95

$= \frac{N_x}{D_x}$ annuity table

Pure Endowment $_nE_x = \frac{D_{x+n}}{D_x}$

L E S S O N XIII.

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It is customary in this country to use a symbol different in some respects from the

$$N_x = D_{x+1} + D_{x+2} + \ldots\ldots\ldots \text{ to end of table,}$$

introduced in the 11th lesson. This symbol is $\mathfrak{N}_x$ (or sometimes $\dot{N}_x$), and its value is defined as follows:--

$$\mathfrak{N}_x = D_x + D_{x+1} + D_{x+2} + \ldots\ldots\ldots \text{ to end of table.}$$

On comparison it will be seen that the only difference lies in the fact that N_x begins with D_{x+1}, while $\mathfrak{N}_x$ begins with D_x. It is easy to establish the relation between N_x and $\mathfrak{N}_x$; for,--

$$\mathfrak{N}_x = D_x + (D_{x+1} + D_{x+2} + \ldots\ldots\ldots \text{ to end of table})$$
$$= D_x + N_x.$$

also $\quad N_{x-1} = D_x + D_{x+1} + \ldots\ldots\ldots \text{ to end of table} = \mathfrak{N}_x$

and $\quad \mathfrak{N}_{x+1} = D_{x+1} + D_{x+2} + \ldots\ldots \text{ to end of table} = N_x.$

Recapitulating, we have:--

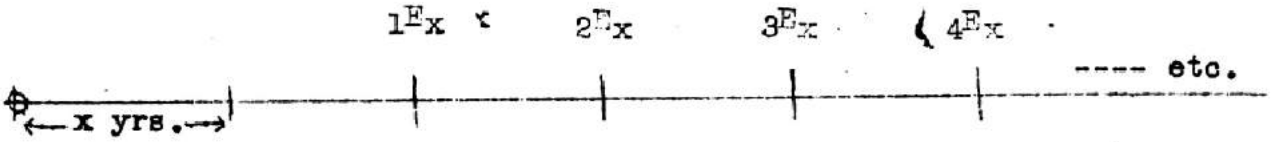
$$N_x = \mathfrak{N}_{x+1} \tag{63}$$
$$\mathfrak{N}_x = N_{x-1} = D_x + N_x.$$

In American tables the column headed N_x is almost certain to be what we have above defined as $\mathfrak{N}_x$; the easiest way to determine this is to look at the last age x for which l_x and hence D_x is not equal to zero. If the number in the "N column" corresponding to this age is the same as D_x, it is an "$\mathfrak{N}$ column," but if it is zero, it is an "N column." The commutation symbol $\mathfrak{N}_x$ will be

found tabulated for the American Experience table at 3% on p. 76. Returning to formula (62), we see that the relations exhibited in (63) enable us to write

$$a_x = \frac{\mathbb{N}_{x+1}}{D_x} = \frac{N_x}{D_x} \tag{64}$$

The formula for a_x may be derived in another way.



If a person aged x should possess a pure endowment of 1 due in 1 year, a pure endowment of 1 due in 2 years, etc., throughout the extent of the table, these would evidently be equivalent to an immediate annuity of 1 for life. Hence we have,--

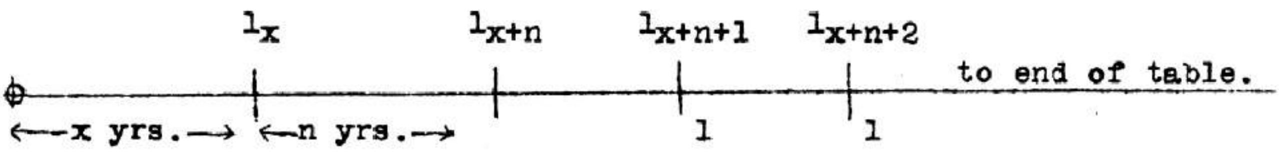
$$a_x = {}_1E_x + {}_2E_x + {}_3E_x + \ldots\ldots \text{to end of table,}$$

$$= \frac{D_{x+1}}{D_x} + \frac{D_{x+2}}{D_x} + \frac{D_{x+3}}{D_x} + \text{etc., to end of table,}$$

by formula (61). Therefore, as before,

$$a_x = \frac{D_{x+1} + D_{x+2} + \ldots \text{ to end of table}}{D_x} = \frac{N_x}{D_x} = \frac{\mathbb{N}_{x+1}}{D_x}$$

We now pass to the consideration of a <u>deferred annuity</u>. Let the annuity be deferred n years so that the first payment occurs at the end of n + 1 years and continues thereafter annually throughout life.



Let each of l_x persons aged x purchase an immediate life annuity deferred n years. The first payment would be due at the end of

n + 1 years and there would be l_{x+n+1} survivors each to receive 1 unit. The present value of l_{x+n+1} at rate of interest i would be $v^{n+1}\ l_{x+n+1}$. One year later there would be l_{x+n+2} survivors, and $v^{n+2}\ l_{x+n+2}$ units invested now at rate of interest i would in n + 2 years amount to l_{x+n+2} units, thus supplying each individual with 1 unit. Proceeding in this way we have the total sum which must be on hand at the present moment:--

$$v^{n+1}\ l_{x+n+1} + v^{n+2}\ l_{x+n+2} + v^{n+3}\ l_{x+n+3} + \text{to end of table.}$$

This sum must be raised by equal contributions on the part of the original l_x purchasers of annuities, hence, denoting the present value of an immediate life annuity on a person of age x deferred n years by ${}_{n|}a_x$ (read "n bar a sub x"), we have:

$$ {}_{n|}a_x = \frac{v^{n+1}\ l_{x+n+1} + v^{n+2}\ l_{x+n+2} + \dots \text{ to end of table}}{l_x}$$

Multiplying both numerator and denominator by v^x we have,--

$$ {}_{n|}a_x = \frac{v^{x+n+1}\ l_{x+n+1} + v^{x+n+2}\ l_{x+n+2} + \dots\dots \text{ to end of table.}}{v^x\ l_x}$$

hence,

$$ {}_{n|}a_x = \frac{D_{x+n+1} + D_{x+n+2} + \dots \text{ to end of table}}{D_x} = \frac{N_{x+n}}{D_x} = \frac{\mathbb{N}_{x+n+1}}{D_x}$$

The formula, then, for an annuity deferred n years is:

$$ {}_{n|}a_x = \frac{N_{x+n}}{D_x} = \frac{\mathbb{N}_{x+n+1}}{D_x} \qquad (65)$$

It will be noticed that when n = 0, that is, when the annuity is deferred zero years, (65) reduces, as it should, to (64).

N O T A T I O N .

We shall make a few remarks at this point concerning Actuarial Notation.

A letter at the lower left corner of the principal symbol relates to the number of years covered by the probability or the benefit in question. Thus,

${}_np_x \equiv$ probability that (x) will live n years,

(x) means "a person aged x,"

$\equiv$ means "identically equals," or "denotes,"

${}_nP_x \equiv$ annual premium, limited to n payments, for a whole life insurance of 1 on (x),

${}_nE_x \equiv$ value of a pure endowment on (x), payable if he survives n years.

If the letter comes <u>before</u> a perpendicular bar $|$, it shows that a <u>deferred</u> period is meant; if after, a temporary period is meant. Thus, ${}_{n|}a_x \equiv$ value of an annuity on (x) deferred n years, i. e., the first payment at end of $\overline{n+1}$ years.

${}_{|n}a_x \equiv$ value of a temporary annuity on (x) for n years,

${}_{n|}q_x \equiv$ probability that (x) will die in a year, deferred n years, i. e., that he will die in the $\overline{n+1}$th year,

${}_{|n}q_x \equiv$ probability that (x) will die within n years,

${}_{n|m}a_x \equiv$ value of a temporary annuity of m years on (x) deferred n years.

Our next problem will be to find the value of a temporary annuity on (x) for n years. Perhaps the easiest way to discover this is to make use of the evident fact that a temporary annuity for n years must be equal to the excess of a life annuity over an annuity deferred n years. Hence,

$$_{|n}a_x = a_x - {}_{n|}a_x$$

or,

$$_{|n}a_x = \frac{N_x}{D_x} - \frac{N_{x+n}}{D_x} = \frac{N_x - N_{x+n}}{D_x} \tag{66}$$

Using the American symbol we have,

$$_{|n}a_x = \frac{\mathbb{N}_{x+1} - \mathbb{N}_{x+n+1}}{D_x} \tag{66a}$$

If we agree to write, as is often done in actuarial works,—

$$N_x - N_{x+n} = N_{x\overline{n|}}$$

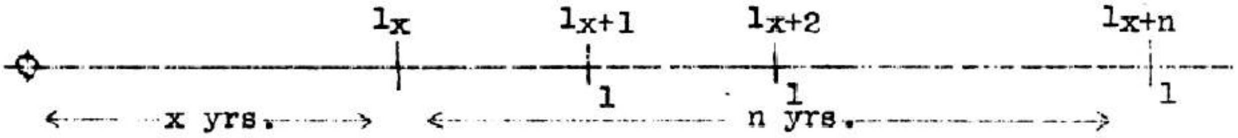
$$\mathbb{N}_x - \mathbb{N}_{x+n} = \mathbb{N}_{x\overline{n|}}$$

$$\mathbb{N}_{x+1} - \mathbb{N}_{x+1+n} = \mathbb{N}_{x+1:\overline{n|}} \qquad \text{etc.,}$$

the above formulae becomes:

$$_{|n}a_x = \frac{N_{x\overline{n|}}}{D_x} = \frac{\mathbb{N}_{x+1:\overline{n|}}}{D_x} \tag{67}$$

The temporary annuity may also be derived independently, as follows:



We have, reasoning as in previous cases:

the present value of l_{x+1} units at i% due in 1 year is $v\, l_{x+1}$,

	do.	l_{x+2}	do.	2 years is $v^2 l_{x+2}$,
	do.	l_{x+3}	do.	3 " " $v^3 l_{x+3}$,
-----	-----	-----	-----	-----
	do.	l_{x+n}	do.	n " " $v^n l_{x+n}$,

hence, summing and dividing by l_x, we have,--

$$|_n a_x = \frac{v\, l_{x+1} + v^2\, l_{x+2} + \ldots\ldots + v^n\, l_{x+n}}{l_x}$$

$$= \frac{v^{x+1}\, l_{x+1} + v^{x+2}\, l_{x+2} + \ldots + v^{x+n}\, l_{x+n}}{v^x\, l_x}$$

$$= \frac{D_{x+1} + D_{x+2} + \ldots\ldots\ldots + D_{x+n}}{D_x}.$$

But, $\quad N_x = D_{x+1} + D_{x+2} + \ldots + D_{x+n} + D_{x+n+1} + \ldots$ to end of table

and $\quad N_{x+n} = \qquad D_{x+n+1} + \ldots$ to end of table

Subtracting

$$N_x - N_{x+n} = D_{x+1} + D_{x+2} + \ldots + D_{x+n}.$$

Therefore, substituting in the above formula,--

$$|_n a_x = \frac{N_x - N_{x+n}}{D_x} = \frac{\mathbb{N}_{x+1} - \mathbb{N}_{x+n+1}}{D_x} = \frac{N_{x\overline{n|}}}{D_x} = \frac{\mathbb{N}_{x+1:\overline{n|}}}{D_x}$$

*1. Find the value of $|_n a_x$ by considering it as a series of pure endowments.

The intercepted, or deferred, temporary annuity is deferred n years and then continues for m years. It is evidently in value equal to the excess of an annuity deferred n years over one deferred n + m years.

Hence,

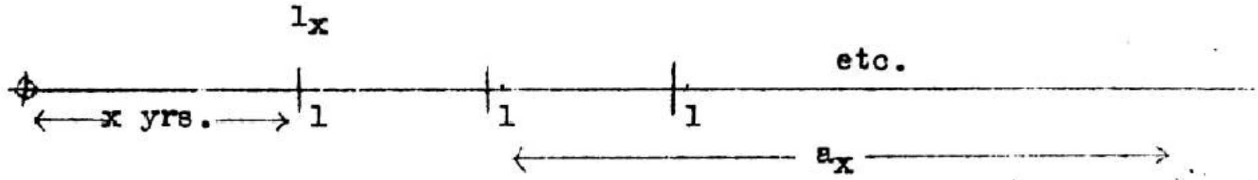
$$_{n|m}a_x = {}_{n|}a_x - {}_{n+m|}a_x$$

$$= \frac{N_{x+n}}{D_x} - \frac{N_{x+n+m}}{D_x} = \frac{N_{x+n:\overline{m|}}}{D_x} \qquad (68)$$

$$= \frac{\mathbb{N}_{x+n+1}}{D_x} - \frac{\mathbb{N}_{x+n+m+1}}{D_x} = \frac{\mathbb{N}_{x+n+1:\overline{m|}}}{D_x} \qquad (68a)$$

*2. Find the value of $_{n|m}a_x$ by two other methods.

When the payments (annual rent) are made at the beginning of each interval we have an annuity-due. The symbol for this is $\mathfrak{a}_x$.



This figure shows that,--

$$\mathfrak{a}_x = 1 + a_x = 1 + \frac{N_x}{D_x} = \frac{D_x + N_x}{D_x}$$

$$\therefore \quad \mathfrak{a}_x = 1 + a_x = \frac{N_{x-1}}{D_x} = \frac{\mathbb{N}_x}{D_x} \qquad (69)$$

It is worth noting here that by formulas (62) and (69),

$$a_x = \frac{N_x}{D_x} \quad \text{and} \quad \mathfrak{a}_x = \frac{\mathbb{N}_x}{D_x},$$

that is, the British N_x divided by D_x gives the value of an annuity-immediate, a_x; while the American $\mathbb{N}_x$ divided by D_x gives the value of an annuity-due, $\mathfrak{a}_x = 1 + a_x$.

*3. Show that,

$$_{n|}\mathfrak{a}_x = {}_{n-1|}a_x = \frac{N_{x+n-1}}{D_x} = \frac{\mathbb{N}_{x+n}}{D_x}$$

*4. Show that,

$$|_n\ddot{a}_x = 1 + |_{n-1}a_x = \frac{N_{x-1} - N_{x+n-1}}{D_x} = \frac{N_{x-1:\overline{n|}}}{D_x}$$

$$= \frac{\mathbb{N}_x - \mathbb{N}_{x+n}}{D_x} = \frac{\mathbb{N}_{x\overline{n|}}}{D_x}.$$

Show that,

$$_{n|m}\ddot{a}_x = {}_{n-1|m}a_x = \frac{N_{x+n-1} - N_{x+n+m-1}}{D_x} = \frac{N_{x+n-1:\overline{m|}}}{D_x}$$

$$= \frac{\mathbb{N}_{x+n} - \mathbb{N}_{x+n+m}}{D_x} = \frac{\mathbb{N}_{x+n:\overline{m|}}}{D_x}$$

Recapitulating, we have;

Immediate at end of yr. | Annuity due at beginning of yr.

$$\begin{aligned} a_x &= \frac{N_x}{D_x} \\ _{n|}a_x &= \frac{N_{x+n}}{D_x} \\ |_n a_x &= \frac{N_{x\overline{n|}}}{D_x} \\ _{n|m}a_x &= \frac{N_{x+n:\overline{m|}}}{D_x} \end{aligned} \qquad (70) \qquad\qquad \begin{aligned} \ddot{a}_x &= \frac{\mathbb{N}_x}{D_x} \\ _{n|}\ddot{a}_x &= \frac{\mathbb{N}_{x+n}}{D_x} \\ |_n\ddot{a}_x &= \frac{\mathbb{N}_{x\overline{n|}}}{D_x} \\ _{n|m}\ddot{a}_x &= \frac{\mathbb{N}_{x+n:\overline{m|}}}{D_x} \end{aligned} \qquad (70a)$$

It is thus seen that the British N is particularly adapted to express the immediate annuity, and the American $\mathbb{N}$ is particularly adapted to express the annuity-due. It is also evident, that if in the formulae for an immediate annuity the British N be exchanged for the American $\mathbb{N}$, they are transformed into the corresponding formulae for an annuity-due, and vice versa.

It may be of some interest to the student to observe by a numerical example the actual working and sufficiency of an annuity

fund as obtained by the preceding analysis. For this purpose we select a group of immediate life annuities of 1 per annum issued to each of 5485 persons aged 85. By formula (64) the value of each annuity is

$$a_{85} = \frac{N_{86}}{D_{85}} = \frac{93.68369}{44.46441} = 2.106936,$$

hence 5485 such annuities would cost

$$2.106936 \times 5485 = 11556.54.$$

Table showing the operation of an annuity fund of $11556.54 formed by a group of 5485 persons of age 85 each purchasing an immediate life annuity of one dollar per annum. Based on American Experience Mortality Table at three per cent.

(1) Age	(2) Year	(3) Fund at commencement of year	(4) Interest earned in year	(5) Sum of (3) and (4)	(6) Claims	(7) Fund at end of year
85	1	11556.54	346.70	11903.24	4193	7710.24
86	2	7710.24	231.31	7941.55	3079	4862.55
87	3	4862.55	145.87	5008.42	2146	2862.42
88	4	2862.42	85.87	2948.29	1402	1546.29
89	5	1546.29	46.39	1592.68	847	745.68
90	6	745.68	22.37	768.05	462	306.05
91	7	306.05	9.18	315.23	216	99.23
92	8	99.23	2.98	102.21	79	23.21
93	9	23.21	.70	23.91	21	2.91
94	10	2.91	.09	3.00	3	0.00

PROBLEMS.

*6. A certain magazine is $4.00 per annum in advance. At what price could the publishing company afford to make its patrons

life subscribers, money being worth 3%: (a) a subscriber aged (x); (b) aged 35?

*7. Each of a large group of pensioners aged 50 receives an annual pension of $360.00. Assuming the American Experience table and 3% interest, what immediate settlement could the government afford to make with each pensioner?

*8. Each of a group of individuals is owing a corporation $846.00, but the debtors are unable to pay the total sum at one time. Money being worth 3%, what annual sum to be paid throughout life could the corporation afford to accept from each individual in lieu of the total debt of $846.00, the first payment to be made at once?

*9. Find the value of:

$$_{30|}a_{20} \qquad _{10|20}a_{30} \qquad _{|16}a_{42} \qquad _{33|23}a_{13}\,.$$

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L E S S O N XIV.

Expectations of Life and Probable Life.

If we could determine the total future-lifetime in years of a large group of l_x persons of present age x, this number, divided by l_x, would be what is termed the expectation of life at age x for that group. It is thus seen that the expectation of life is another name for the average future lifetime, or, as it is sometimes called, the "average after-lifetime." Although we cannot read the future and know exactly when each individual life will end and thus determine the total future lifetime of the group, still we may assume with a fair degree of certainty that the death rate in the group will be well represented by some standard table of mortality based on observations of lives in that region or territory. The expectation of life, therefore, is first of all based upon some selected mortality table; it will be found that for different mortality tables the expectation of life at a given age is not the same, although the range of variation is inconsiderable. For example, we have:

Expectation	of	Life	at	age	35	by H^M (British) table is	31.05	yrs.
"	"	"	"	"	35	" Carlisle table is	31.00	"
"	"	"	"	"	35	" Amer. Experience table is	31.78	"
"	"	"	"	"	35	" Actuaries' table is	30.87	"
"	"	"	"	"	35	" 23 German Offices table is	29.16	"

In order to derive from a given mortality table what is technically known as the expectation of life of a person aged x it is necessary to make some assumption regarding the distribution of deaths within each year. Among the various assumptions that might be made we shall consider only the following: (a) that all the deaths occur at the beginning of the year, (b) that all the deaths occur at the end of the year, (c) that the deaths occur uniformly within the year at the middle of intervals in number equal to the number of deaths.

(a) Deaths occur at beginning of year.

This assumption leads to what has been termed the "curtate expectation of life" which is denoted by the symbol e_x.

Living	l_x $\uparrow$	l_{x+1} living through 1st future year	l_{x+2} living through 2d future year	etc.
Dying	$\downarrow$ d_x		$\downarrow$ d_{x+1}	$\downarrow$ d_{x+2}

The diagram shows that out of $l_x = d_x + l_{x+1}$ persons of age x, d_x die at the beginning of the year, and the remaining l_{x+1} survive one year and so, as a whole, the original l_x lives complete exactly l_{x+1} years of life in the first future year. Similarly the $l_{x+1} = d_{x+1} + l_{x+2}$ survivors are divided into two groups, d_{x+1} who die at the beginning of the second future year and l_{x+2} who continue to live to the end of that year, attaining the age x + 2; thus the group of l_{x+1} lives completes in this second future year a total of l_{x+2} years of life. Proceeding in this way we see that the total number of future years lived is

$$l_{x+1} + l_{x+2} + l_{x+3} + l_{x+4} + \ldots\ldots \text{ to end of table.}$$

Dividing this by l_x we have the average future lifetime or curtate expectation of life of each individual given by the formula

$$e_x = \frac{l_{x+1} + l_{x+2} + l_{x+3} + l_{x+4} + \dots \text{ to end of table}}{l_x}. \quad (71)$$

Hence the rule to find the Curtate Expectation of Life: divide the sum of all those living above the given age by the number living at the given age. For example, we have (see Am. Ex. table, p. 75),

$$e_{90} = \frac{462 + 216 + 79 + 21 + 3}{847} = \frac{781}{847} = .92$$

An inspection of the developments on p. 72 will show that e_x may be regarded as the present value of an immediate life annuity when the rate of interest $i = 0$, for then the discount factor

$$v = \frac{1}{1 + i} = \frac{1}{1 + 0} = 1,$$

and the expression for a_x there obtained reduces to that above given by formula (71).

(b) Deaths occur at end of year.

Living l_x

↑ l_x living through | l_{x+1} living through | etc.

1st future year ↓ 2d future year ↓

Dying d_x d_{x+1}

As all deaths take place at the end of the year each of the initial l_x persons completes one year of life in the first future year and the group therefore aggregates l_x years of life in this year. Then d_x lives come to an end and l_{x+1} lives continue through the second future year thus completing l_{x+1} more full years of life. At the end of the second year d_{x+1} deaths take

place leaving l_{x+2} lives to continue and survive the third future year, etc. In this manner we arrive at the following total future lifetime for the group:

$$l_x + l_{x+1} + l_{x+2} + l_{x+3} + \ldots\ldots \text{ to end of table.}$$

Hence the average future lifetime or expectation of life according to hypothesis (b) is

$$\frac{l_x + l_{x+1} + l_{x+2} + l_{x+3} + l_{x+4} + \ldots \text{ to end of table}}{l_x}$$

$$= 1 + \frac{l_{x+1} + l_{x+2} + l_{x+3} + l_{x+4} + \ldots \text{ to end of table}}{l_x}$$

$$= 1 + e_x. \qquad (72)$$

It is thus seen that the expectation derived by this hypothesis exceeds the curtate expectation by one year. As the value of an annuity-due exceeds that of the annuity-immediate by 1 it follows from the preceding reasoning that the expectation $1 + e_x$ is the present value of a life annuity-due at age x when interest is zero. We may thus by analogy in a sense regard the expectations obtained by assumptions (a) and (b) respectively as the "expectation-immediate" and "expectation-due". Since the expectation-immediate assumes all deaths at the beginning of the year and the expectation-due assumes all deaths at the end of the year it would be reasonable to suppose that the mean of these two expectations would give a value of the expectation of life more nearly in accordance with the facts. The mean value is $\frac{1}{2}(e_x + 1 + e_x) = \frac{1}{2} + e_x$. This is called the <u>complete expectation of life</u> and is denoted by the symbol $\mathring{e}_x$. We have therefore the following formula:

$$\overset{\circ}{e}_x = \frac{1}{2} + e_x \tag{73}$$

which shows that the complete expectation of life always exceeds the curtate expectation of life by 1/2. Formula (73) in connection with the rule before given for finding the curtate expectation of life leads to the following rule:

To find the Complete Expectation of Life, $\overset{\circ}{e}_x$, divide the sum of all those living above the given age by the number living at the given age and add 1/2 to the result.

For example, we have (see Am. Ex. table, p. 75),

$$\overset{\circ}{e}_{85} = \frac{4193 + 3079 + 2146 + 1402 + 847 + 462 + 216 + 79 + 21 + 3}{5485} + \frac{1}{2}$$

$$= \frac{12448}{5485} + \frac{1}{2} = 2.27 + 0.50 = 2.77.$$

P r o b l e m s.

*Example 1. Prove that $e_x = p_x(1 + e_{x+1})$.

*Example 2. What do you understand by the temporary (for n years) curtate expectation of life? Show that

$$|_n e_x = \frac{l_{x+1} + l_{x+2} + l_{x+3} + l_{x+4} + \ldots\ldots\ldots\ldots + l_{x+n}}{l_x}$$

*Example 3. State what is meant by a deferred (for n years) curtate expectation of life and prove that that

$$_{n|}e_x = \frac{l_{x+n+1} + l_{x+n+2} + \ldots\ldots\ldots\ldots \text{ to end of table}}{l_x}$$

*Example 4. Prove that $e_x = |_n e_x + {}_{n|}e_x$.

*Example 5. Prove that

$$e_x = \frac{N_x}{D_x} @ 0\% \text{ and } 1 + e_x = \frac{\mathbb{N}_x}{D_x} @ 0\%.$$

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LESSON XV.

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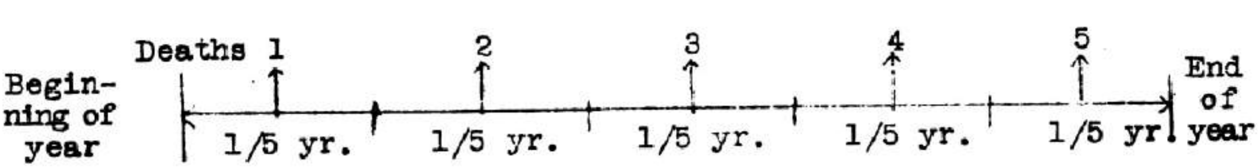
Expectation of Life and Probable Lifetime (continued).

We shall now study assumption (c) of the preceding lesson and show that it leads to what was there defined as the complete expectation of life, $\mathring{e}_x$. According to this hypothesis the deaths occur within the year at the middle of intervals in number equal to the number of deaths. For example, if there are twelve deaths in the year they will occur at the middle of each month, if 365 deaths take place in the year they will occur at noon each day. A careful examination of this distribution of deaths discloses that the total number of years of life lived in the year of death by all the persons who die within the year is just one half their number.

Deaths 1 2 3 4 5 6 7 8 9 10 11 12
Month Jan. Feb. Mar. Apr. May June July Aug. Sep. Oct. Nov. Dec.

For example, the 12 persons who die in the middle of each month of the year aggregate $\frac{12}{2}$ = 6 years of life in that year for while the first lives 1/2 month the last lives one year lacking 1/2 month and the two together live a full year, similarly with the remaining five pairs. In the case of an odd number, as 5, dying within the year



we see that 1 and 5 together live a year, 2 and 4 together live a year, while 3 lives 1/2 year, dying in the middle of the year.

Thus again the total years lived in the year of death is one-half the number of deaths, or 5/2.

According to the mortality table, in a group of l_x persons of age x, d_x die in the first future year, d_{x+1} in the second future year, d_{x+2} in the third future year, etc. Consequently, in connection with the above remarks, the following diagram shows:

Living at age x	l_x ↑		
	d_x die in	d_{x+1} die in	etc.
	1st future year	2d future year	

that the d_x persons dying in the first future year live $\frac{1}{2}d_x$ years of life in the first future year, that the d_{x+1} persons dying in the second future year live $\frac{1}{2}d_{x+1}$ years of life in the second future year plus d_{x+1} years of life in the preceding year, that the d_{x+2} persons dying in the third future year live $\frac{1}{2}d_{x+2}$ years of life in the third future year plus $2d_{x+2}$ years of life in the two preceding years, etc., thus leading to the following schedule:

(1) Future year	(2) Deaths in year	(3) Total years lived in year of death	(4) Total years lived in years preceding year of death and after age x	(5) Total years lived after age x = (3) + (4)
First	d_x	$\frac{1}{2}d_x$	0	$\frac{1}{2}d_x$
Second	d_{x+1}	$\frac{1}{2}d_{x+1}$	d_{x+1}	$\frac{3}{2}d_{x+1}$
Third	d_{x+2}	$\frac{1}{2}d_{x+2}$	$2d_{x+2}$	$\frac{5}{2}d_{x+2}$
Fourth	d_{x+3}	$\frac{1}{2}d_{x+3}$	$3d_{x+3}$	$\frac{7}{2}d_{x+3}$
etc.....to end of table			etc.....to end of table.	

Hence, adding column (5), the total number of future years lived is:

$$\frac{1}{2}d_x + \frac{3}{2}d_{x+1} + \frac{5}{2}d_{x+2} + \frac{7}{2}d_{x+3} + \ldots\ldots \text{to end of table,}$$

or, breaking the sum up into the portions found by adding columns (3) and (4) respectively, we have

$$\begin{aligned}
&\frac{1}{2}d_x + \frac{1}{2}d_{x+1} + \frac{1}{2}d_{x+2} + \frac{1}{2}d_{x+3} + \ldots\ldots \text{to end of table} \\
&\qquad + d_{x+1} + 2d_{x+2} + 3d_{x+3} + \ldots\ldots \qquad \text{do.} \\
= &\frac{1}{2}(d_x + d_{x+1} + d_{x+2} + d_{x+3} + \ldots\ldots \text{to end of table)} \\
&\qquad + d_{x+1} + d_{x+2} + d_{x+3} + \ldots\ldots \qquad \text{do.} \\
&\qquad\qquad + d_{x+2} + d_{x+3} + \ldots\ldots \qquad \text{do.} \\
&\qquad\qquad\qquad + d_{x+3} + \ldots\ldots \qquad \text{do.} \\
= &\frac{1}{2}l_x + l_{x+1} + l_{x+2} + l_{x+3} + l_{x+4} + \ldots\ldots \text{to end of table}
\end{aligned}$$

since, by the method of constructing the mortality table,

$$\Sigma d_x = l_x, \ \Sigma d_{x+1} = l_{x+1}, \ \Sigma d_{x+2} = l_{x+2}, \text{ etc.}$$

Hence the average future lifetime is:

$$\frac{1}{2} + \frac{l_{x+1} + l_{x+2} + l_{x+3} + l_{x+4} + \ldots\ldots \text{to end of table}}{l_x}$$

$$= \frac{1}{2} + e_x = \mathring{e}_x.$$

From what has gone before it is clear that the expectation of life is <u>not</u> a term of years which a person may logically <u>expect</u> to live; some will fall short of while others will overreach the term of expectancy. It is merely a kind of average obtained by evening up the excess of years of life in some persons with the defect in others.

The curtate expectation of life may be exhibited as the sum of a series of probabilities of life. Referring back to formula (58) it was there shown that p_x (or better, ${}_1p_x$, employing the notation explained on p. 89) = l_{x+1}/l_x is the probability that (x) will survive one year. By the same method of reasoning ${}_2p_x = l_{x+2}/l_x$ is the probability that (x) will survive 2 years, etc., and ${}_np_x = l_{x+n}/l_x$ is the probability that (x) will survive n years. Denoting the last age in the mortality table by w (to avoid the frequent repetition in our formulas of the phrase, "...... to end of table") we have

$$e_x = \frac{l_{x+1} + l_{x+2} + \ldots\ldots + l_w}{l_x}$$

$$= \frac{l_{x+1}}{l_x} + \frac{l_{x+2}}{l_x} + \ldots\ldots + \frac{l_w}{l_x}$$

$$= {}_1p_x + {}_2p_x + \ldots\ldots + {}_rp_x$$

$$\text{if} \quad r = w - x,$$

r therefore being the remaining number of years in the mortality table after the age x. The above result may be compactly written

$$e_x = \sum_{n=1}^{n=r} {}_np_x$$

which means "the sum of such terms as ${}_np_x$ where n is given in succession all integral values from 1 to r inclusive. For example, for the American Experience table of mortality, w = 95 and if x = 35, since r = w - x = 95 - 35 = 60, we have

$$e_{35} = \sum_{n=1}^{n=60} {}_np_{35} = {}_1p_{35} + {}_2p_{35} + \ldots. + {}_{60}p_{35}.$$

Similarly,

$$e_{90} = \sum_{n=1}^{n=5} {}_nP_{90} = {}_1P_{90} + {}_2P_{90} + {}_3P_{90} + {}_4P_{90} + {}_5P_{90}.$$

Probable Lifetime (Vie Probable).

Another expression sometimes employed in speaking of the future years of an individual is his "Probable Lifetime," or, as the French say, "Vie Probable." This period is defined, in connection with a group of persons all of the same age, as the number of years after which just one half of the members of the group will be alive. The chances therefore are even that a person will survive his term of probable life. In symbols, if

$$\frac{1}{2}l_x = l_{x+t},$$

then t = the term of probable life of a person aged x. For example, by the American Experience table of mortality, of 100000 at age 10 there will be 50000 living sometime between ages 64 and 65; to find the fractional part of the year after which exactly 50000 are living (on the supposition that deaths occur uniformly throughout the year) we have:

$$\begin{aligned} l_{10} &= 100000 \\ l_{64} &= 51230 \\ \tfrac{1}{2}l_{10} &= \underline{50000} \\ \text{difference} &= 1230 \\ d_{64} &= 1889 \end{aligned}$$

Hence the fractional term $= \frac{1230}{1889} = .65$, and at age 64.65 just one-half of those living at age 10 are alive; this gives 64.65 - 10 = 54.65 years as the probable lifetime at age 10.

Hence, $\therefore \quad t_{10} = 54.65$ years

For age 35 the computation would be as follows:

$$l_{35} = 81822$$

$$l_{68} = 43133$$

$$\frac{1}{2}l_{35} = \underline{40911}$$

$$\text{difference} = 2222$$

$$d_{68} = 2243$$

$$\text{fractional term} = \frac{2222}{2243} = .99$$

$$\text{therefore } t_{35} = 68.99 - 35 = 33.99 \text{ years.}$$

*Problem.--Compute a table of <u>Probable Lifetime</u> = t_x for all ages after ten by American Experience mortality table, carrying the fractional period to two places of decimals as in above illustration. Also state for what ages:

(a) the probable life is greater than the complete expectation of life

(b) " " " equals " " " " "

(c) " " " is less than " " " " "

or, in symbols, when

(a) $t_x > \mathring{e}_x$, (b) $t_x = \mathring{e}_x$, (c) $t_x < \mathring{e}_x$.

The sign $>$ means and is read "is greater than," the sign $<$, "is less than." Thus $a > b$ means and is read "a is greater than b."

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L E S S O N XVI.

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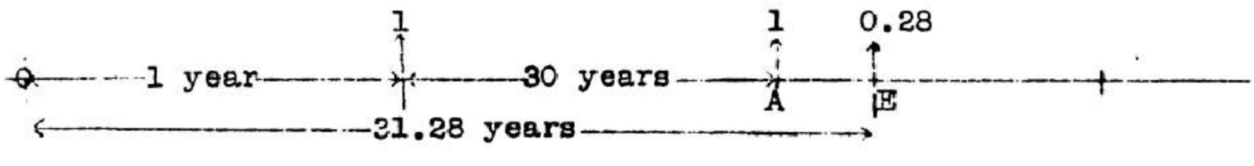
Expectations of Life and Probable Life (continued).

Still another expression met with in actuarial parlance is "The most Probable Lifetime." This, with respect to a group of persons all of the same age, is defined as the difference between the present age and that year of age in which the maximum number of deaths occur in the group. The term "most probable" comes from the assumption that since all the members of the group are on an equal footing any individual of the group may regard that year when the greatest number of deaths takes place as the one in which he is most likely to die. For example, an inspection of the d_x or death column of the American Experience table of mortality shows that a maximum death frequency is reached at about age 73, there being 2505 deaths in that year. The most probable lifetime of a person aged x is therefore 73 - x; for age 35 this gives 38 years as the most probable lifetime.

The following table serves to show the difference between the future terms of life obtained by the several foregoing methods, also the duration of life, or age at death, in each case, all for age 35.

Description	Future term of life	Duration of life
e_{35} = Curtate expectation of life........	31.28	66.28
$\mathring{e}_{35}$ = Complete expectation of life........	31.78	66.78
t_{35} = Probable lifetime................	33.99	68.99
Most probable life time..................	38.00	73.00

These future terms of life are of more than ordinary interest because they are sometimes employed (although not always correctly) to measure the pecuniary loss or damage sustained in case of death or disability due to negligence, for example, death on a railway car due to carelessness of company. Having determined the surplus earning power of the decedent, which may be regarded as a measure of the annual loss to his dependents, this sum is usually taken as the annual rent of an immediate annuity which is to run for a term certain or for life. When a term certain is employed it is frequently the complete expectation of life $\mathring{e}_x$, or the probable life t_x; it is easy to see, however, that these measures are quite arbitrary, and may be wide of the mark, especially if it can be shown that the earning power of the decedent is likely to be variable or cease entirely at a certain age. In this connection it would be interesting to know whether an immediate annuity-certain for the term of the curtate expectation of life is greater, equal to, or less than an immediate life annuity. We shall first investigate this question for the age 35. It has already been shown that e_{35} = 31.28 years so our problem is to find the value of the immediate annuity-certain $a_{\overline{31.28}|}$ and the immediate life annuity a_{35}. To find $a_{\overline{31.28}|}$:



the usual practice is to make (a) the annual payment of 1 at end of each year during 31 years and (b) the fractional payment of 0.28 at the end of 31.28 years. The present value of (a) is

$a_{\overline{31}|}$; the present value of (b) is determined by finding the value of 0.28 at the beginning of the year in which it is due, point A in diagram, at simple interest, and then reducing this amount to the present moment, point O in diagram, at compound interest by multiplying by v^{31}. Using formula (2), simple interest, the value of 0.28 due at E, reduced to point A, the beginning of the year, is at 3%

$$\frac{0.28}{1 + 0.28 \times .03} = \frac{0.28}{1.0084} = .2777,$$

1 + ni

which, multiplied by v^{31} = .39999 gives .1111 as the present value. By the table on page 32 we have at 3%

	(a)	$a_{\overline{31}|}$	= 20.0004
	(b)		= .1111
whence, adding,		$a_{\overline{31.28}|}$	= 20.1115

By referring to the table on page 84 we find the value of a life annuity-due, with payments at the <u>beginning</u> of the year, issued to a person aged 35 is 19.9174; therefore, since a life annuity-immediate, with payments at the <u>end</u> of the year, falls short of the life annuity-due only by the <u>first</u> payment of 1, it is evident that a_{35} = 18.9174. Accordingly it appears that the present value of an immediate annuity-certain for the term of the curtate expectation of life at age 35 <u>exceeds</u> the present value of an immediate life annuity issued to a person aged 35 by 1.1941. To illustrate, if the damage to be awarded were based on an annual loss of $1000, the adjustment by means of the curtate expectation of life would exceed that by the life annuity to the extent of $1194.10, for

1000 $a_{\overline{31.28}\rvert}$	=	20111.50
1000 a	=	18917.40
difference	=	1194.10

Since the terms of the complete expectation of life and probable life are longer than the term of the curtate expectation of life an adjustment on the basis of a term certain equal to either one would be still more favorable to the beneficiaries of the decedent. We have here considered the case of age 35; it may be of interest to know if the relation

$$a_{\overline{e_x}\rvert} > e_x$$

holds for all ages. For those who are mathematically inclined we give below a proof that <u>the present value of an immediate annuity-certain for the term of the curtate expectation of life at any age is always greater than the present value of an immediate life annuity issued at the same age.</u> Since the complete expectation of life exceeds e_x by 1/2 it follows with still greater force that

$$a_{\overline{\mathring{e}_x}\rvert} > a_x$$

at all ages, that is, for all values of x.

<u>Proof.</u>

Let $e_x = n + f$, where n is the integral and f the fractional or decimal part of the number expressing the term. Then, by the preceding lesson, page 104,

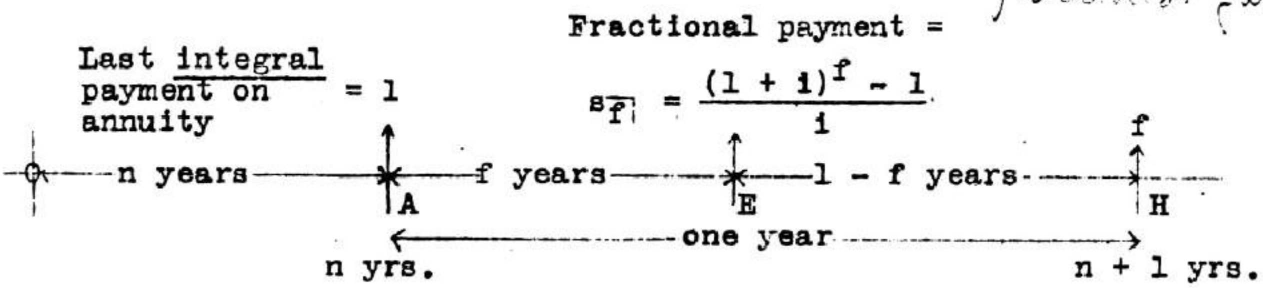
$e_x = n + f = {}_1p_x + {}_2p_x + \ldots\ldots + {}_np_x + {}_{n+1}p_x + {}_{n+2}p_x + \ldots\ldots + {}_rp_x,$
consequently, transferring the terms up to and including ${}_{n+1}p_x$ to the left hand member, we have

$$n + f - {}_1p_x - {}_2p_x - \cdots\cdots - {}_np_x - {}_{n+1}p_x = {}_{n+2}p_x + \cdots\cdots + {}_rp_x,$$

a relation which we shall make use of later in the proof.

In theory, the fractional payment would be made at the end of n + f years

f denotes fractional part of yr.



and, since it is the accumulation or amount of an immediate annuity-certain for the fractional term of f years, would by formula (33) be equal to

$$s_{\overline{f}|} = \frac{(1 + i)^f - 1}{i}$$

The present value of $s_{\overline{f}|}$ due at the end of n + f years, point E in diagram, is found by multiplying this amount by the discount factor v^{n+f}, hence the total present value of the annuity, including the n integral payments made at the end of each year during the first n years, is:

$$a_{\overline{\mathring{e}_x}|} = a_{\overline{n+f}|} = a_{\overline{n}|} + v^{n+f}.\frac{(1 + i)^f - 1}{i}.$$

The next step in the proof will be to show that the present value of the fractional payment $s_{\overline{f}|}$ made after n + f years at point E is greater than the present value of a fractional payment of f made at the end of n + 1 years at point H, or, in algebraic symbols, that:

$$v^{n+f} . s_{\overline{f}|} > v^{n+1} . f$$

To this end we have:

$\left(\frac{1}{1+i}\right)^f$

$$v^{n+f}.s_{\overline{f|}} = v^{n+f}.\ \frac{(1+i)^f - 1}{i} = v^n.\ \frac{v^f(1+i)^f - v^f}{i}$$

$$= v^n\ .\ v\ .\ \frac{1 - v^f}{iv} = v^{n+1}\ .\ \frac{\frac{1}{v} - \frac{v^f}{v}}{i} = v^{n+1}\ .\ \frac{(1+i) - v^{f-1}}{i},$$

hence $$v^{n+f}\ .\ s_{\overline{f|}} = v^{n+1}\ .\ \left[\frac{(1+i) - (1+i)^{1-f}}{i}\right]$$

from which it appears that if we can show that the quantity within the bracket <u>is greater than f</u> the truth of the preceding inequality is established. In order to show that the quantity in the bracket is greater than f we shall find it necessary at this point to expand $(1+i)^{1-f}$ in a series proceeding according to ascending powers of i. This is easily accomplished with the aid of the binomial theorem, concerning which the student is advised to consult any good college algebra. The binomial formula is:

$$(a+b)^m = a^m + \frac{m}{1}.a^{m-1}.b + \frac{m(m-1)}{1.2}.a^{m-2}.b^2 + \frac{m(m-1)(m-2)}{1.2.3}.a^{m-3}.b^3$$

$$+ \frac{m(m-1)(m-2)(m-3)}{1.2.3.4}.a^{m-4}.b^4 + \ldots\ldots\text{etc.}$$

accordingly, setting

$$a = 1, \qquad b = i, \qquad m = 1-f, \qquad \text{we have}$$

$$(1+i)^{1-f} = 1 + \frac{1-f}{1}.i + \frac{(1-f)(1-f-1)}{1.2}.i^2 + \frac{(1-f)(1-f-1)(1-f-2)}{1.2.3}.i^3$$

$$+\ldots\ldots\text{etc.}$$

$$= 1 + i - fi - \frac{(1-f)f}{1.2}.i^2 + \frac{(1-f)f(f+1)}{1.2.3}.i^3 + \ldots\ldots\text{etc.}$$

Transferring 1 + i to the left hand member, changing the signs

of all the terms in both members and dividing both members by i,

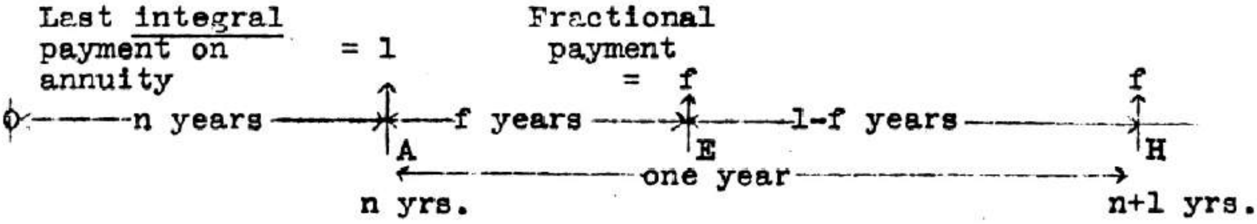
$$\frac{(1 + i) - (1 + i)^{1-f}}{i} = f + \frac{(1-f)f}{1\cdot 2}.i - \frac{(1-f)f(f+1)}{1\cdot 2\cdot 3}.i^2 + \ldots \text{etc.}$$

Enclosing the terms in the series in the right hand member, after the first, in pairs by means of braces, we have

$$\frac{(1 + i) - (1 + i)^{1-f}}{i} = f + \frac{(1-f)f}{1\cdot 2}\left[1 - \frac{f+1}{3}.i\right].i + \ldots\ldots \text{etc.}$$

from which we conclude that the left hand member is greater than f since it is equal to f plus further terms of an infinite series.

In practice, a fractional payment f would be made at the end of n + f years.



It is self evident that the present value of a sum f due at the end of n + f years, point E in diagram, exceeds the present value of the same sum f due 1 - f years later, that is, at the end of n + 1 years, point H in diagram. It appears, then, from what has gone before, that the present value of the fractional payment, due at the end of n + f years, of an immediate annuity-certain, both in theory and practice, exceeds the present value of a sum f due at the end of n + 1 years, point H in both diagrams. Hence in all cases $a_{\overline{e_x}|} = a_{\overline{n+f}|} > a_{\overline{n}|} + v^{n+1}.f$

therefore, since $a_{\overline{n}|} = v + v^2 + v^3 + \ldots\ldots + v^n$

$$a_{\overline{e_x}|} > v + v^2 + v^3 + \ldots\ldots + v^n + v^{n+1} \cdot f \,.$$

Also, referring back to page 72, we find for the immediate life annuity the expression:

$$a_x = v\,\frac{l_{x+1}}{l_x} + v^2\frac{l_{x+2}}{l_x} + \ldots + v^{n+1}\,\frac{l_{x+n+1}}{l_x} + v^{n+2}\,\frac{l_{x+n+2}}{l_x} + \ldots + v^r\,\frac{l_{x+r}}{l_x}.$$

But $\frac{l_{x+1}}{l_x} = {}_1p_x$, $\frac{l_{x+2}}{l_x} = {}_2p_x$, ... $\frac{l_{x+n}}{l_x} = {}_np_x$, etc.,

accordingly, we may write,

$$a_x = v\,{}_1p_x + v^2\,{}_2p_x + \ldots + v^{n+1}\,{}_{n+1}p_x + v^{n+2}\,{}_{n+2}p_x + \ldots + v^r\,{}_rp_x \,.$$

Subtracting this equation from the above inequality, we have:

$$\begin{aligned}
a_{\overline{e_x}|} - a_x &= v(1 - {}_1p_x) + \ldots\ldots + v^n(1 - {}_np_x) + v^{n+1}(f - {}_{n+1}p_x) \\
&\qquad - v^{n+2}\,{}_{n+2}p_x - \ldots\ldots\ldots\ldots - v^r\,{}_rp_x \\
&> v^{n+1}(1 - {}_1p_x + \ldots\ldots + 1 - {}_np_x + f - {}_{n+1}p_x) \\
&\qquad - v^{n+2}\,{}_{n+2}p_x - \ldots\ldots\ldots\ldots - v^r\,{}_rp_x \\
&= v^{n+1}(n + f - {}_1p_x - \ldots\ldots\ldots - {}_np_x - {}_{n+1}p_x) \\
&\qquad - v^{n+2}\,{}_{n+2}p_x - \ldots\ldots\ldots\ldots - v^r\,{}_rp_x \\
&= v^{n+1}({}_{n+2}p_x + \ldots\ldots\ldots\ldots\ldots + {}_rp_x) \\
&\qquad - v^{n+2}\,{}_{n+2}p_x - \ldots\ldots\ldots\ldots - v^r\,{}_rp_x
\end{aligned}$$

on account of the relation established at the top of page 111. The above inequalities will give the student no trouble if he will

remember that v raised to any power is <u>greater</u> than v raised to a still higher power, since v, the discount factor, is always less than 1. The last of these inequalities may be written:

$$a_{\overline{e_x}|} - a_x > v^{n+1}\left[(1 - v)\,{}_{n+2}p_x + (1 - v^2)\,{}_{n+3}p_x + \ldots + (1 - v^{r-n-1})\,{}_{r}p_x\right]$$

the right hand member of which is evidently made up of positive terms and hence must be greater than zero. Therefore

$$a_{\overline{e_x}|} - a_x > 0, \quad \text{that is,} \quad a_{\overline{e_x}|} > a_x, \text{ which was to be proved.}$$

*Example 1. Find the present value of $s_{\overline{f}|}$ on page 112 by first accumulating $s_{\overline{f}|}$ to the end of n + 1 years, point H, and then reducing or discounting this amount n + 1 years to the point O.

Review Problems and Exercises.

*Example 1. Assuming that a purchaser desires to realize interest at the rate i, investigate a formula for determining the price to be paid for a bond of 1 securing an annual dividend of j and redeemable at a premium n years hence.

$100 irredeemable stock bearing an annual dividend of $4 was bought 20 years ago for $90. Find at what figure it must now be sold in order that the vendor may realize 5 per cent. per annum on the transaction. Institute of Actuaries Examination, Part II., April, 1903.

*Example 2. Show how an annuity at any age may be expressed in terms of that at the next higher age. Actuarial Society of America, Examination, Part I., May, 1901.

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James J. Glover.

L E S S O N XVII.

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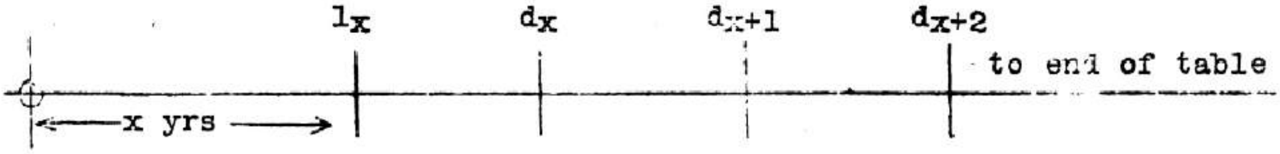
To find the value of an insurance for life of 1 on (x):



Let each individual of a group of l_x persons, each of age x, contract to take an insurance of 1. During the first year d_x of these individuals will die, hence the sum of d_x units will mature at the end of the first year. The value of this sum at the beginning of the year is vd_x, interest being at the rate i. After two years, d_{x+1} further contracts will mature owing to the death of the d_{x+1} individuals out of the l_{x+1} starting out at the beginning of the second year. The present value of d_{x+1} units due two years hence is v^2d_{x+1}. Proceeding in this way we have the following scheme:

$v\,d_x$	is the present value of	d_x	units due in	one	year,
v^2d_{x+1}	Do.	d_{x+1}	Do.	two	years,
v^3d_{x+2}	Do.	d_{x+2}	Do.	three	years,

etc., to end of table.

Whence, adding, there must be in hand at the present time:

$$vd_x + v^2d_{x+1} + v^3d_{x+2} + \ldots\ldots\ldots\ldots \text{ to end of table,}$$

units in order to meet contracts as they mature. It must be borne in mind that this sum must be immediately invested so as to yield

at least i% interest, compounded annually. Since each of the l_x members who start out ought to share the same portion of the total present value, we have:

$$A_x = \frac{v\,d_x + v^2 d_{x+1} + v^3 d_{x+2} + \dots \text{ to end of table}}{l_x}$$

as the contribution due from each individual. This formula may be modified somewhat as follows:--multiply numerator and denominator by v^x, then

$$A_x = \frac{v^{x+1} d_x + v^{x+2} d_{x+1} + v^{x+3} d_{x+2} + \dots \text{ to end of table}}{v^x l_x}$$

We now introduce a new commutation symbol, viz.: $C_x = v^{x+1} d_x$; hence,

$$\left.\begin{aligned} C_x &= v^{x+1}\, d_x \\ C_{x+1} &= v^{x+2}\, d_{x+1} \\ C_{x+2} &= v^{x+3}\, d_{x+2} \\ &\text{etc.,} \\ C_{x+n} &= v^{x+n+1}\, d_{x+n} \end{aligned}\right\} \qquad (74)$$

and the preceding formula may be written,--

$$A_x = \frac{C_x + C_{x+1} + C_{x+2} + \dots \text{ to end of table}}{D_x} \qquad (75)$$

Finally, introducing another commutation symbol, M_x, which we define as follows:

$$M_x = C_x + C_{x+1} + C_{x+2} + \dots \text{ to end of table,} \qquad (76)$$

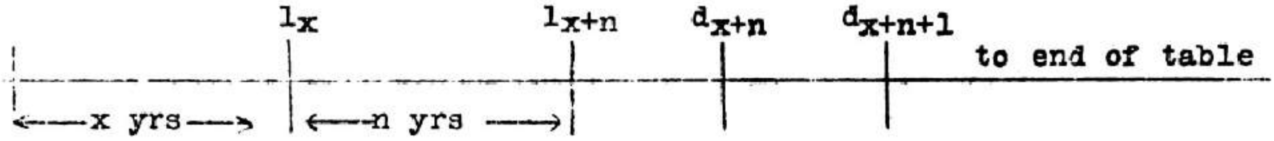
formula (75) takes the simple and well-known form:

$$A_x = \frac{M_x}{D_x} \qquad (77)$$

The single premium for an ordinary life insurance of 1, viz.:

A_x, is thus easily expressed in terms of commutation symbols. This premium, A_x, it must be understood, is a net premium. The commutation symbols, C_x and M_x, will be found tabulated for the American Experience table at 3% on pages 80 and 76.

We shall next seek the value of an insurance of 1, issued to (x), deferred n years. The symbol for this is $_{n|}A_x$.



Referring to the figure we see that after n years l_{x+n} survive of the l_x who started at age x. By the Mortality Table, d_{x+n} lives out of l_{x+n} fail during the (n + 1)th year after age x. Therefore n + 1 years hence, d_{x+n} units will mature, and the present value of d_{x+n} units due in n + 1 years is $v^{n+1} d_{x+n}$. In this manner we arrive at the scheme:

$v^{n+1}d_{x+n}$	is the present value of	d_{x+n}	units due in	n + 1	years,
$v^{n+2}d_{x+n+1}$	do.	d_{x+n+1}	do.	n + 2	"
$v^{n+3}d_{x+n+2}$	do.	d_{x+n+2}	do.	n + 3	"

etc., to the end of the table.

The total present sum required to meet the maturing contracts is therefore,

$$v^{n+1}d_{x+n} + v^{n+2}d_{x+n+1} + v^{n+3}d_{x+n+2} + \ldots\ldots \text{ to end of table.}$$

Dividing this charge equally among the l_x members who purchase the deferred insurance, we have,

$$_{n|}A_x = \frac{v^{n+1}d_{x+n} + v^{n+2}d_{x+n+1} + v^{n+3}d_{x+n+2} + \ldots \text{ to end of table}}{l_x}$$

To reduce this formula to commutation symbols we multiply, as usual, both numerator and denominator by v^x, and have,--

$$_{n|}A_x = \frac{v^{x+n+1}d_{x+n} + v^{x+n+2}d_{x+n+1} + v^{x+n+3}d_{x+n+2} + \text{.. to end of table}}{v^x\, l_x}$$

Whence, by formula (74),

$$_{n|}A_x = \frac{C_{x+n} + C_{x+n+1} + C_{x+n+2} + \text{... to end of table}}{D_x} \qquad (78)$$

and, by (76),--

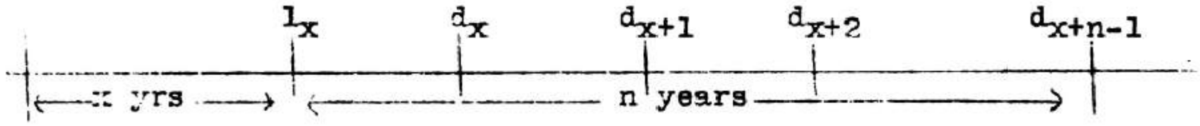
$$_{n|}A_x = \frac{M_{x+n}}{D_x} \qquad (79)$$

Our next problem will be to determine the value of a temporary insurance of 1, for n years, issued to (x). Perhaps the simplest method of solution is to make use of the evident fact that a temporary insurance of n years is equal to a whole life insurance less an insurance deferred n years. This gives:

$$_{|n}A_x = A_x - {_{n|}A_x} = \frac{M_x}{D_x} - \frac{M_{x+n}}{D_x}$$

therefore, $$_{|n}A_x = \frac{M_x - M_{x+n}}{D_x} = \frac{M_{x\overline{n}|}}{D_x} \qquad (80)$$

This formula can be derived in the same manner as the ordinary whole life and deferred life insurance formulas:



Here we have,

$v\,d_x$	is the present value of	d_x	due in one	year,
$v^2 d_{x+1}$	do.	d_{x+1}	" " two	years,
$v^3 d_{x+2}$	do.	d_{x+2}	" " three	"
	etc., to			
$v^n d_{x+n-1}$	do.	d_{x+n-1}	" " n	"

Adding, and dividing by l_x, we have,--

$$|_n A_x = \frac{v d_x + v^2 d_{x+1} + v^3 d_{x+2} + \ldots\ldots + v^n d_{x+n-1}}{l_x}$$

and multiplying numerator and denominator by v^x,

$$|_n A_x = \frac{v^{x+1} d_x + v^{x+2} d_{x+1} + \ldots\ldots + v^{x+n} d_{x+n-1}}{v^x\, l_x}$$

$$= \frac{C_x + C_{x+1} + \ldots\ldots + C_{x+n-1}}{D_x}$$

Now,

$M_x \;= C_x + \ldots\ldots + C_{x+n-1} + C_{x+n} + C_{x+n+1} + \ldots\ldots$ to end of table,

and,

$M_{x+n} = \qquad\qquad C_{x+n} + C_{x+n+1} + \ldots\ldots$ to end of table,

Hence, $M_x - M_{x+n} = C_x + \ldots\ldots + C_{x+n-1}$, and our formula becomes,

$$|_n A_x = \frac{M_x - M_{x+n}}{D_x} \quad \text{as before.}$$

We may also have a deferred temporary, or as it is sometimes called, an intercepted, insurance. Let the insurance of 1, issued to (x), be deferred n years and then temporary m years. In value it is clearly equal to the excess of an insurance deferred n years over an insurance deferred (n+m) years.

Hence, $_{n|m}A_x = {}_{n|}A_x - {}_{n+m|}A_x = \frac{M_{x+n}}{D_x} - \frac{M_{x+n+m}}{D_x}$

and, $$_{n|m}A_x = \frac{M_{x+n} - M_{x+n+m}}{D_x} = \frac{M_{x+n:\overline{m|}}}{D_x} \qquad (81)$$

*It is left as an exercise for the student to derive formula (81) by the other method, using diagram to illustrate.

A very simple relation exists between the annuity a_x and the insurance A_x; we now proceed to derive it.

$$C_x = v^{x+1}d_x = v^{x+1}(l_x - l_{x+1}) = v^{x+1}l_x - v^{x+1}l_{x+1}$$
$$= v.v^x l_x - v^{x+1}l_{x+1} = vD_x - D_{x+1}\ .$$

Therefore,
$$C_x = vD_x - D_{x+1}$$
$$C_{x+1} = vD_{x+1} - D_{x+2} \qquad (82)$$
$$C_{x+2} = vD_{x+2} - D_{x+3}$$

etc., to end of table.

Adding, $\Sigma C_x = v\Sigma D_x - \Sigma D_{x+1}$.

Here the sign Σ (the Greek capital letter Sigma) is a symbol of summation, used for purposes of abbreviation; thus,--

$\Sigma C_x = C_x + C_{x+1} + C_{x+2} + \ldots\ldots\ldots$ to end of table,

$\Sigma D_x = D_x + D_{x+1} + D_{x+2} + \ldots\ldots\ldots$ to end of table,

$\Sigma D_{x+1} = D_{x+1} + D_{x+2} + \ldots\ldots\ldots$ to end of table.

But, $\Sigma C_x = M_x$; $\Sigma D_x = \aleph_x$; $\Sigma D_{x+1} = N_x$;

and the above formula becomes,--

$$M_x = v\aleph_x - N_x. \qquad (83)$$

Dividing both sides of (83) by D_x, we have,--

$$\frac{M_x}{D_x} = v\frac{\aleph_x}{D_x} - \frac{N_x}{D_x}\ .$$

which, by formulas (77), (70a) and (70), may be written,

$$A_x = v\ddot{a}_x - a_x, \text{ whence, by (69),}$$

$$A_x = v(1 + a_x) - a_x. \tag{84}$$

Formula (84) expresses the relation which invariably exists between the quantities A_x and a_x. It shows that if either one is given the other can be readily determined. Remembering that,

$$v = \frac{1}{1 + i}, \quad d = iv = 1 - v, \text{ etc.,}$$

formula (84) may be thrown into several other useful and interesting forms. For example,

$$A_x = v(1 + a_x) - a_x = \frac{1 + a_x}{1 + i} - a_x$$

$$= \frac{1 + a_x - a_x - ia_x}{1 + i}$$

Hence, $$A_x = \frac{1 - ia_x}{1 + i} \tag{85}$$

*It is left for the student to show that:

$$A_x = v(1 - ia_x) \tag{86}$$

$$= v - (1 - v)a_x \tag{87}$$

$$= 1 - d(1 + a_x) \tag{88}$$

*Example.--Prove that,

$$_{n|}A_x = v\ _{n|}\ddot{a}_x - {}_{n|}a_x.$$

Below are two numerical schedules drawn to illustrate the sufficiency of the single premiums found by the use of formulas (77) and (80). By formula (77) the net value of a whole life insurance for 1 issued to a person aged 85 is:

$$A_{85} = \frac{M_{85}}{D_{85}} = \frac{40.440681}{44.46441} = .9095066,$$

hence the fund due to the contributions of 5485 members purchasing

such insurances is:

$$5485 \times .9095066 = 4988.644.$$

Table showing the operation of an insurance fund of $4988.644 formed by a group of 5485 persons of age 85 each purchasing a whole life insurance of $1 payable at end of year of death. Based on American Experience mortality table with interest at three per cent.

(1) Age	(2) Year	(3) Fund at commencement of year	(4) Interest earned in year	(5) Sum of (3)+(4)	(6) Death claims	(7) Fund at end of year	(8) Survivors at end of year	(9) Reserve, or policy value = (7)÷(8)
85	1	4988.644	149.659	5138.303	1292	3846.303	4193	.91731
86	2	3846.303	115.389	3961.692	1114	2847.692	3079	.92488
87	3	2847.692	85.431	2933.123	933	2000.123	2146	.93202
88	4	2000.123	60.004	2060.127	744	1316.127	1402	.93875
89	5	1316.127	39.484	1355.611	555	800.611	847	.94523
90	6	800.611	24.018	824.629	385	439.629	462	.95158
91	7	439.629	13.189	452.818	246	206.818	216	.95749
92	8	206.818	6.205	213.023	137	76.023	79	.96232
93	9	76.023	2.281	78.304	58	20.304	21	.96686
94	10	20.304	.609	20.913	18	2.913	3	.97100
95	11	2.913	.087	3.000	3	.000	0	

By formula (80) the net value of a temporary, or term, insurance for 1 issued to a person aged 20 to extend over a period of 5 years is:

$$|_5A_{20} = \frac{M_{20} - M_{25}}{D_{20}} = \frac{1697.40776 - 1514.57076}{5129.087} = \frac{182.837}{5129.087} = .0356471,$$

hence the fund due to the contribution of 92637 members purchasing

such insurances is:

$$92637 \times .0356471 = 3302.240.$$

Table showing the operation of an insurance fund of \$3302.240 formed by a group of 92637 persons of age 20 each purchasing a temporary, or term, life insurance of \$1 payable at the end of year of death in case death occurs within five years. Based on American Experience mortality table with interest at three per cent.

(1) Age	(2) Year	(3) Fund at commencement of year	(4) Interest earned in year	(5) Sum of (3)+(4)	(6) Death claims	(7) Fund at end of year	(8) Survivors at end of year	(9) Reserve, or policy value = (7)÷(8)
20	1	3302.240	99.067	3401.307	723	2678.307	91914	.02914
21	2	2678.307	80.349	2758.656	722	2036.656	91192	.02233
22	3	2036.656	61.100	2097.756	721	1376.756	90471	.01522
23	4	1376.756	41.303	1418.059	720	698.059	89751	.00778
24	5	698.059	20.941	719.000	719	000.000	89032	.00000

*Example 1. Find the value of A_{35}, $_{5|}A_{35}$, $_{|5}A_{35}$, $_{10|5}A_{35}$.

*Example 2. Derive a numerical schedule illustrating the operation of an insurance fund formed by a group of 31822 persons age 35 each effecting a temporary (for five years) insurance of \$1.

*Example 3. Derive a numerical schedule illustrating the operation of an insurance fund formed by a group of 81322 persons age 35 each effecting a deferred temporary (deferred ten years, then temporary five years) insurance of \$1.

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LESSON XVIII.

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In the preceding lesson an algebraic relation was shown to exist between the insurance A_x and the annuity a_x, namely,

$$A_x = v.\ddot{a}_x - a_x = v(1 + a_x) - a_x$$

It was deduced by an analytic process, and then by means of algebraic transformations it was thrown into several other forms, as (85), (86), (87), (88).

We shall now show how some of these formulas may be otherwise obtained and verbally interpreted. Perhaps the formula (85),

$$A_x = \frac{1 - ia_x}{1 + i}$$

admits of the simplest derivation. An amount 1 deposited by (x) will produce during life an annual interest i payable at the end of each year to the insured. At the end of the year of death there will be the original unit 1 and the interest i for that year payable to the beneficiary, that is, an insurance of 1 + i. Hence a payment of 1 will furnish to (x) a life annuity of annual rent i and a life insurance of 1 + i. The value of the annuity is ia_x, and the value of the insurance is $(1 + i)A_x$. Therefore,

$$1 = ia_x + (1 + i)A_x$$

whence, $(1 + i)A_x = 1 - ia_x$

and $$A_x = \frac{1 - ia_x}{1 + i}$$

The formula at the head of this lesson admits of a direct verbal interpretation. It shows that the value, A_x, of an

insurance of 1 on (x) is the excess of an annuity-due, with annual rent v, that is, $v\ddot{a}_x = v(1 + a_x)$, over an annuity-immediate, with annual rent 1, that is, a_x. This is evidently the case, for consider any year which (x) <u>enters upon</u> and <u>completes</u>:

```
              Contribution of the annuity-      (1+i)v = 1
          v = due at the beginning of the   to  at end of
Beginning ↑   year which accumulates            the year  | End of
----------o-------Year entered upon and completed---------o--------
 of year  |                   Contribution of the         |  year
                              annuity-immediate at = 1
                              the end of the year
```

~~Here~~ the annuity-due will furnish the rent v at the <u>beginning</u> of the year which at interest during the year will accumulate to $(1 + i)v = 1$ at the <u>end</u> of the year; while the annuity-immediate with rent 1 will furnish 1 at the <u>end</u> of the year. Thus at the <u>end</u> of the year the excess of the contribution of the former over the latter is nil. It is different, however, in the case of the year of death, that is, the year <u>entered upon but not completed</u>. Here it is the same as before with the annuity-due, but the annuity-immediate yields nothing; hence there is an excess of 1 at the <u>end</u> of the year of death. This amount would go to the beneficiary and may be regarded as an insurance of 1 on (x); whence,

$$A_x = v\ddot{a}_x - a_x.$$

*Example 1.--Prove that $A_x = v - da_x$ where $d = i/(1+i)$, and give verbal interpretation after the manner of the illustrations in the first part of this lesson.

*Example 2.--Give verbal interpretation of formula (86).

*Example 3.--Give verbal interpretation of formula (87).

*Example 4.--Give verbal interpretation of formula (88).

We have been considering formulas in which A_x is expressed in terms of a_x. It is an easy matter to derive a_x in terms of A_x from these same formulas. For example:

$$A_x = v(1 + a_x) - a_x$$

$$A_x = v + va_x - a_x$$

Transposing, $\quad a_x - va_x = v - A_x$

hence, $\quad a_x = \dfrac{v - A_x}{1 - v}$

and since $\quad 1 - v = 1 - \dfrac{1}{1 + i} = \dfrac{i}{1 + i} = d$

therefore, $\quad a_x = \dfrac{v - A_x}{d} \qquad (89)$

*Example 5.--Prove that $\quad a_x = \dfrac{1 - (1 + i)A_x}{i} \qquad (90)$

*Example 6.--Prove that $a_x = \dfrac{1 - A_x}{d} - 1 \qquad (91)$

Perpetuities.

An annuity of which the payments are to continue forever is called a perpetuity. A perpetuity may be an immediate perpetuity, a perpetuity-due, a deferred perpetuity, etc., the significance attaching to these terms being similar to that in the case of annuities-certain heretofore dealt with. It is clear that the perpetuity is a form of annuity-certain, $a_{\overline{n}|}$, where n, the term, is infinite ($n = \infty$). For this reason the symbol used to denote a perpetuity, with annual rent 1, is a_{∞}. If the annual rent 1 is paid in instalments of 1/m at the end of each mth part of the year, the symbol used is $a_{\infty}^{(m)}$.

Thus:

Immediate 1 1 ------forever.
Perpetuity = a_{∞} 1 year 2 years

Immediate 1/3 1/3 1/3 1/3 1/3 1/3 ------forever.
Perpetuity = $a_{\infty}^{(3)}$ 1 year 2 years

It is easy to see that the present value of the perpetuity a_{∞} at the effective rate of interest i is,--

$$a_{\infty} = \frac{1}{i} \tag{92}$$

for $1/i$, invested at interest at rate i, would yield an annual interest of $i \times 1/i = 1$, that is, a perpetual income of 1 per annum.

Similarly, the present value of the perpetuity $a_{\infty}^{(m)}$ at the nominal rate of interest j, convertible m times per annum, is,--

$$a_{\infty}^{(m)} \text{ at } j_{(m)}\% = \frac{1}{j} \tag{93}$$

for, since the nominal rate is j per annum, it would be j/m for the interval of $1/m$th of a year, and the interest yield of the principal $1/j$ would be $1/j \times j/m = 1/m$, that is, a perpetual income of $1/m$ at the end of every mth part of a year, thus making an annual rent of $m \times 1/m = 1$.

The result (92) may be established by referring back to the formula for $a_{\overline{n}|}$ derived in the theory of annuities-certain. Thus we had, formula (30),

$$a_{\overline{n}|} = \frac{1 - \left[\frac{1}{1+i}\right]^n}{i}$$

and if the term be made infinite, that is, $n = \infty$, in this

expression, it becomes:

$$a_{\infty} = \frac{1 - \left[\frac{1}{1+i}\right]^{\infty}}{i} = \frac{1-0}{i} = \frac{1}{i}$$

for $\frac{1}{1+i}$ is less than unity, and if such a quantity be multiplied by itself an infinite number of times the product will diminish indefinitely and approach zero as a limit.

Similarly, it may easily be shown that the immediate annuity-certain, with annual rent 1, payable uniformly in m equal instalments of 1/m, has a present value:

$$a_{\overline{n}|}^{(m)} = \frac{1}{m} \cdot \frac{1 - \frac{1}{(1 + j/m)^{mn}}}{j/m} = \frac{1 - \frac{1}{(1 + j/m)^{mn}}}{j}$$

when the term-certain is n years and the nominal rate of interest is j convertible m times per annum. *This is left as an exercise for the student. If, now, in this formula the term-certain be increased indefinitely, $n = \infty$, then it becomes:

$$a_{\infty}^{(m)} = \frac{1 - \frac{1}{(1 + j/m)^{\infty}}}{j} = \frac{1-0}{j} = \frac{1}{j}$$

Having now explained what is meant by the perpetuity a_{∞}, we return to our discussion at the head of this lesson and add thereto the following formulas:

$$A_x = \frac{a_{\infty} - a_x}{1 + a_{\infty}} \qquad (94)$$

$$a_x = a_{\infty} - (1 + a_{\infty})A_x \qquad (95)$$

These formulas, with the aid of (92), may be at once derived from those at the end of the preceding lesson. Formula (94) may also

be derived and interpreted verbally, an exercise which the student should carry out.

*Example 7. Show that,--

$$a_{\infty} = a_{\overline{n}|} + {}_{n|}a_{\infty}$$

$$\text{and } \ddot{a}_{\infty} = 1 + a_{\infty}$$

Review Problems and Exercises.

*Example 1. Prove mathematically that $a_{\overline{e_x}|} > a_x$. Institute of Actuaries Examination, Part II., April, 1903.

*Example 2. Find the compound interest on one dollar for three-fourths of a year, (a) when the effective rate is 4 per cent.; when the nominal rate is 4 per cent. and convertible (b) yearly, (c) half yearly. Act. Soc. of Amer., Exam., Part I., May, 1901.

*Example 3. Given a table of mortality, show how to find (a) at what age it is most probable a person of a given age will die; (b) how many years he has an even chance of living. Act. Soc. of Amer., Exam., Part I., May, 1901.

*Example 4. Assuming interest at 5 per cent., the present value of an annuity of $1, payable at the end of the year on a given life, is found to be $11.66. What is the present value of an insurance of $1 on the same life? Act. Soc. of Amer., Exam., Part I., June, 1900.

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